Projection

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Projection

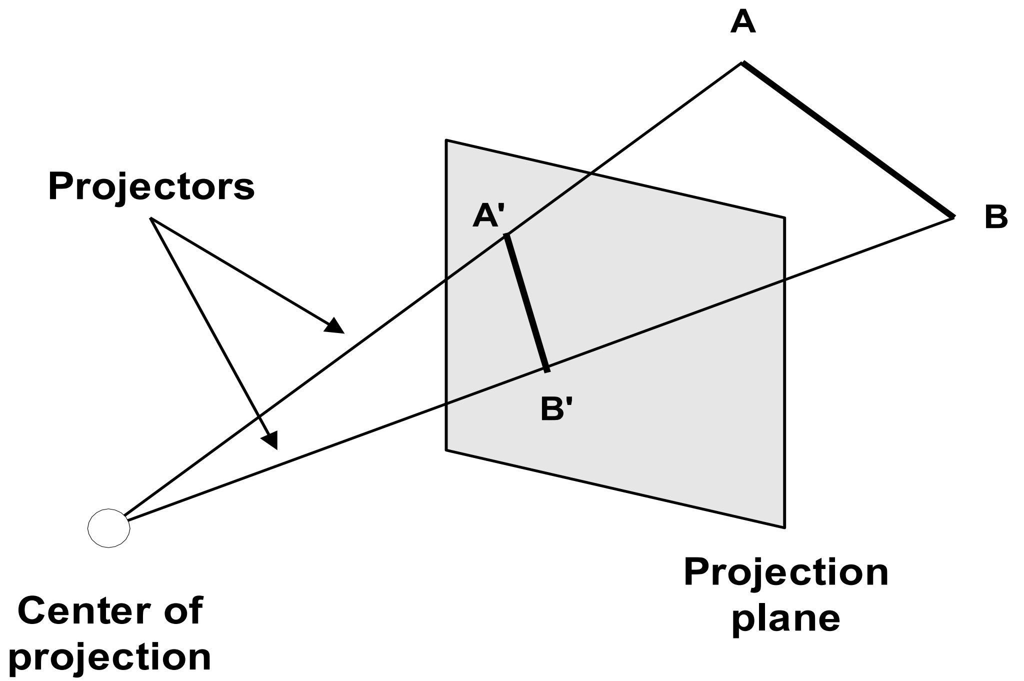
- In general, ***projections*** transform points in a coordinate system of dimension *n* into points in a coordinate system of dimension less than *n*.

- We shall limit ourselves to the projection from 3D to 2D.

- In computer graphics
  - Map viewing coordinates to 2D screen coordinates

- We will deal with ***planar geometric projections*** where:
  - The projection is onto a plane rather than a curved surface
  - The projectors are straight lines rather than curves

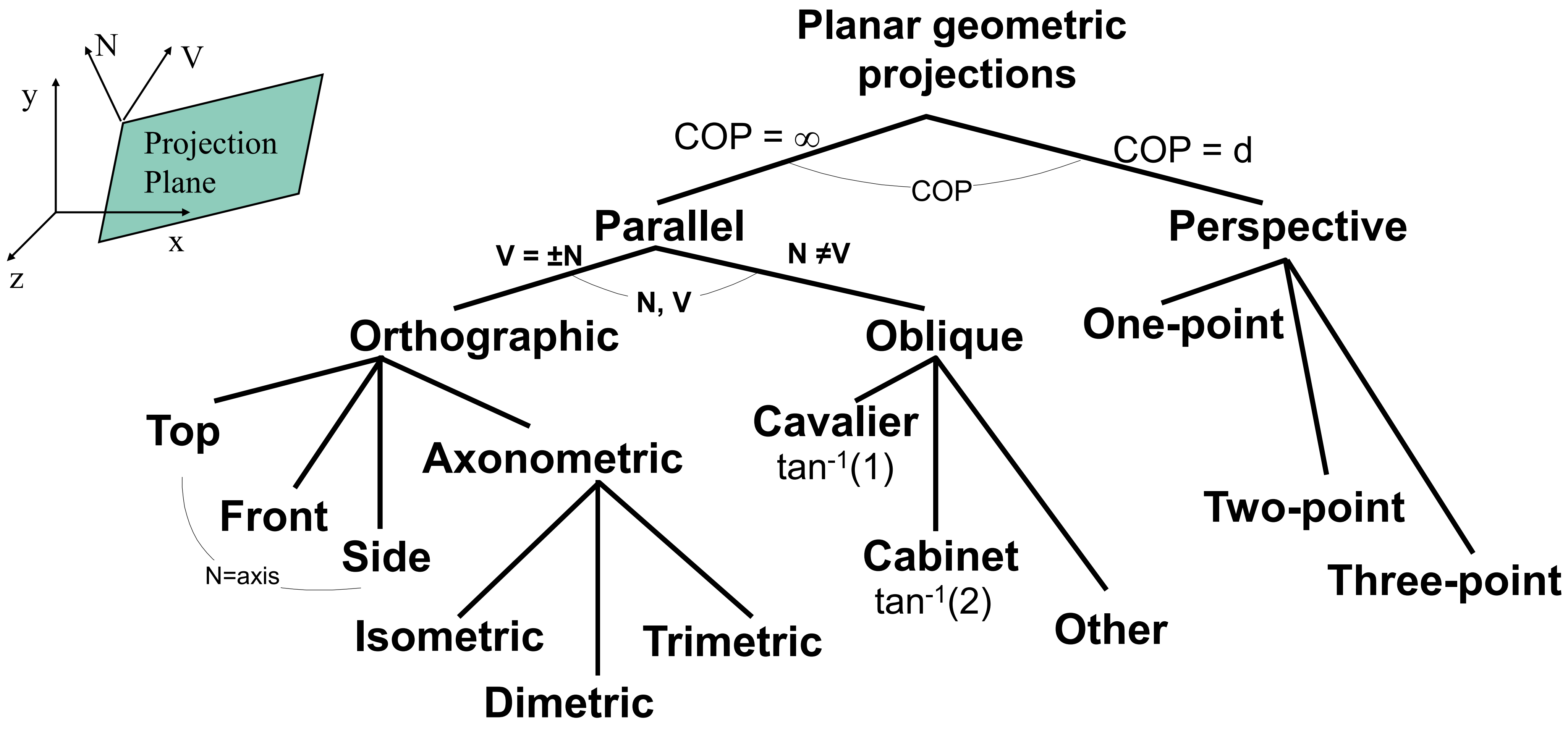
Projections – key terms



A (3D) → A’ (2D)

- The ***projection*** of a <u>3D object</u> is defined by <u>straight projection rays</u> (called ***projectors***) emanating from a ***center of projection***, passing through each point of the object, and intersecting a ***projection plane*** to form the projection.

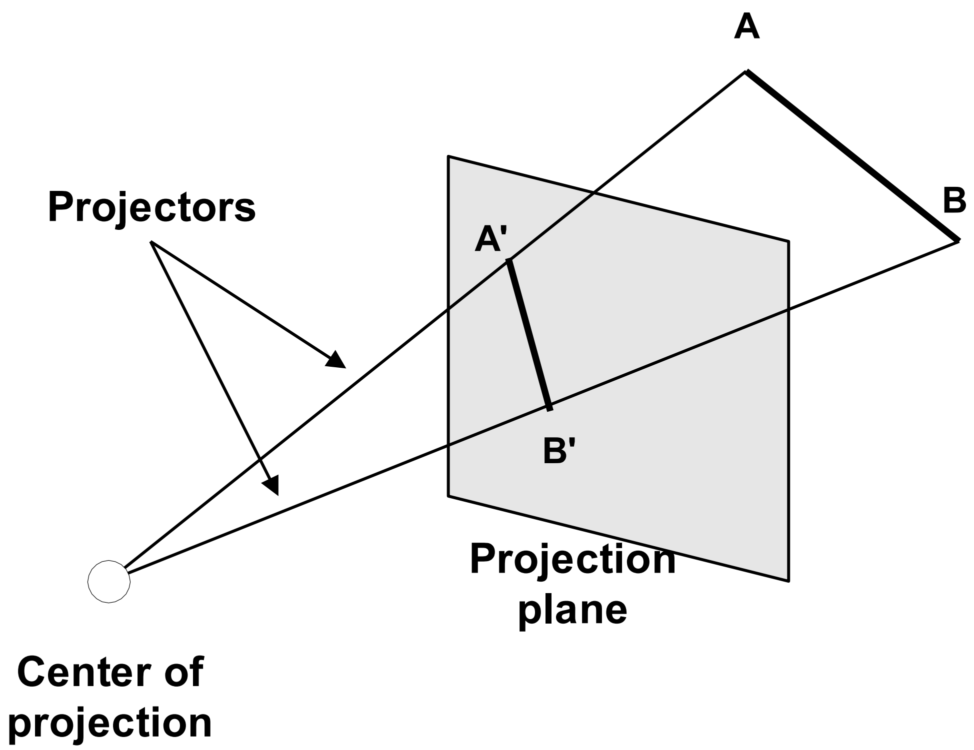
Planner Geometric Projections taxonomy



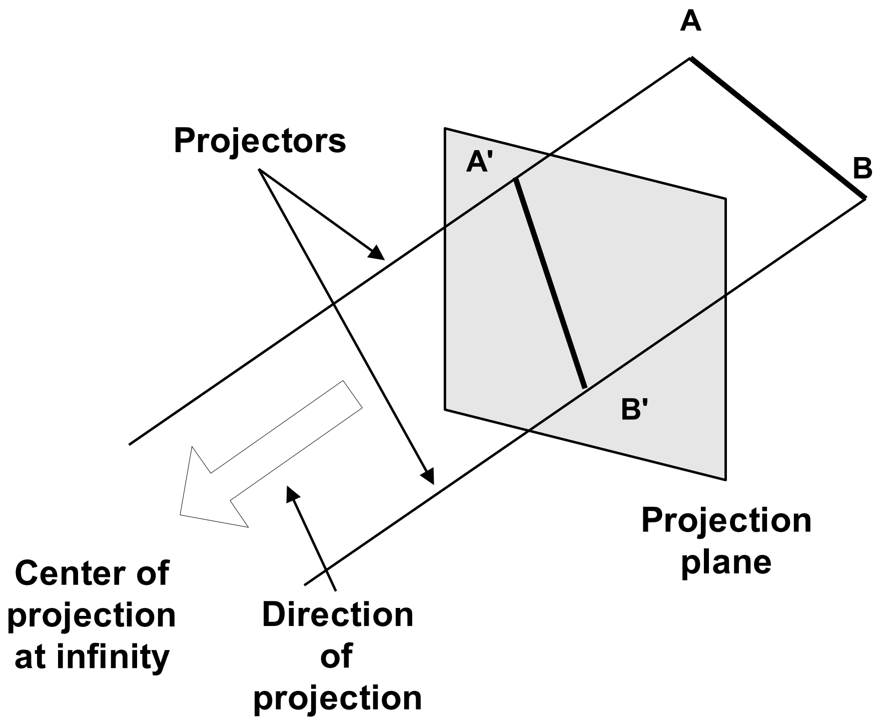
Subclasses of planar geometric projections

Planer Projection – Major Types

- Key factor is the ***<u>center of projection, COP</u>***.
  - if distance to center of projection is finite : **Perspective**
  - if infinite : **Parallel** -> so needs direction of projection vector, DOP



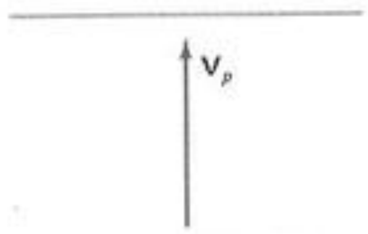
Perspective projection



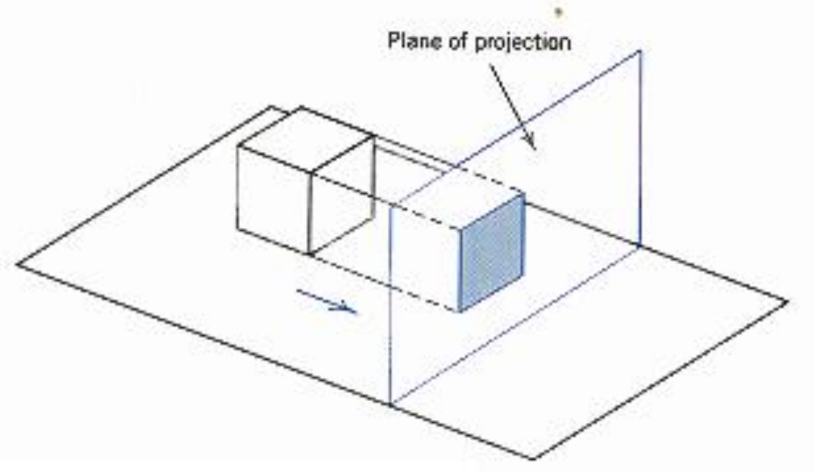
Parallel projection

Parallel projection - Types

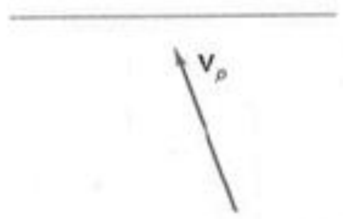
- **Orthographic** projection
  - the projection is **perpendicular** to the view plane



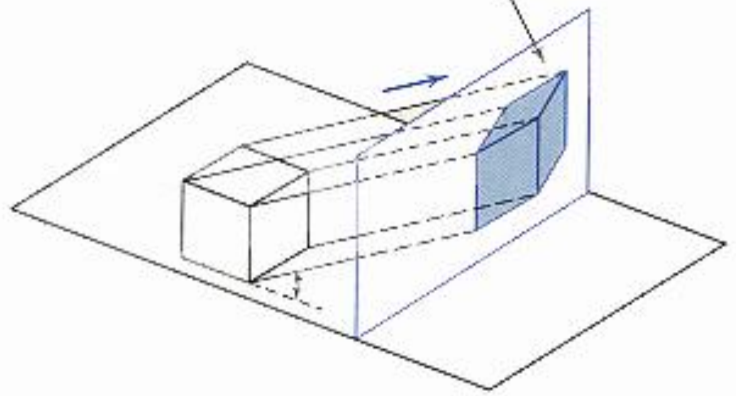
Orthographic Projection



- **Oblique** projection
  - The projectors are **inclined** with respect to the view plane



Oblique Projection

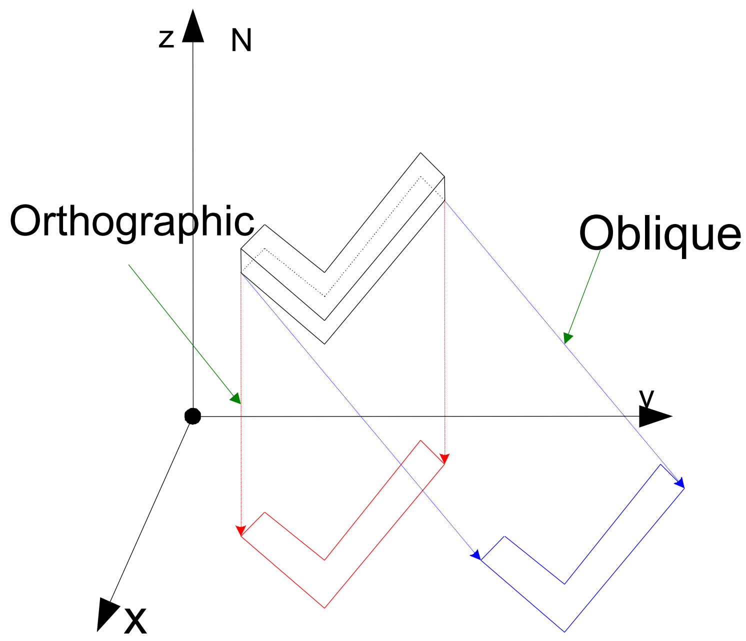


Parallel projection - Types

- 2 principle types:
  - on the basis of ***DOP, V*** and ***projection plane normal N***)

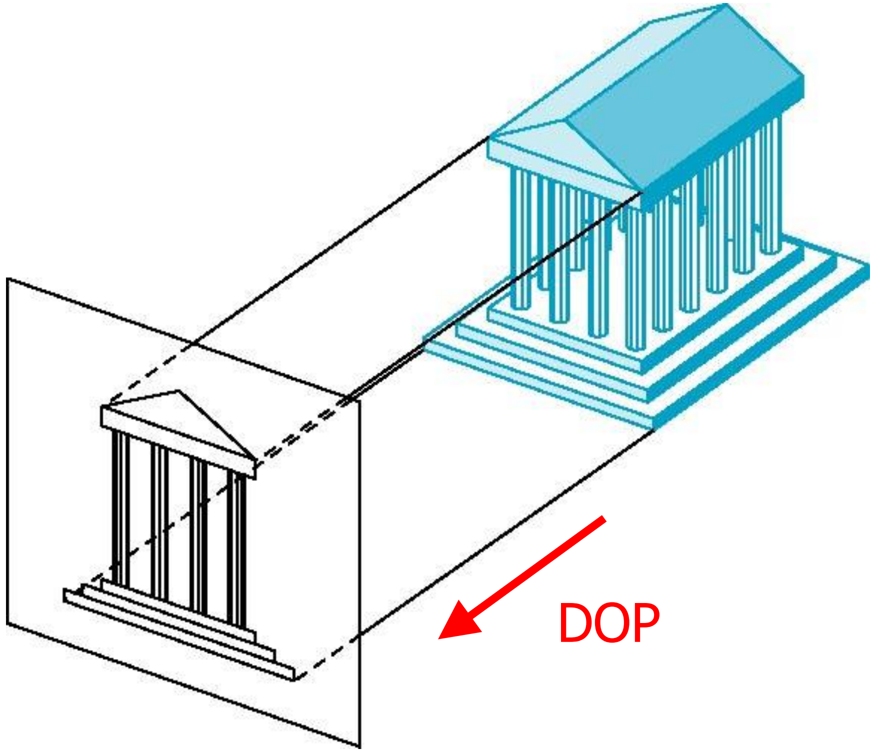
- Orthographic :
  - ***V*** and ***N*** are the same or the reverse of each other, i.e. V is perpendicular to view plane

- Oblique :
  - direction of projection **!=** the projection plane normal.



Orthographic Projection

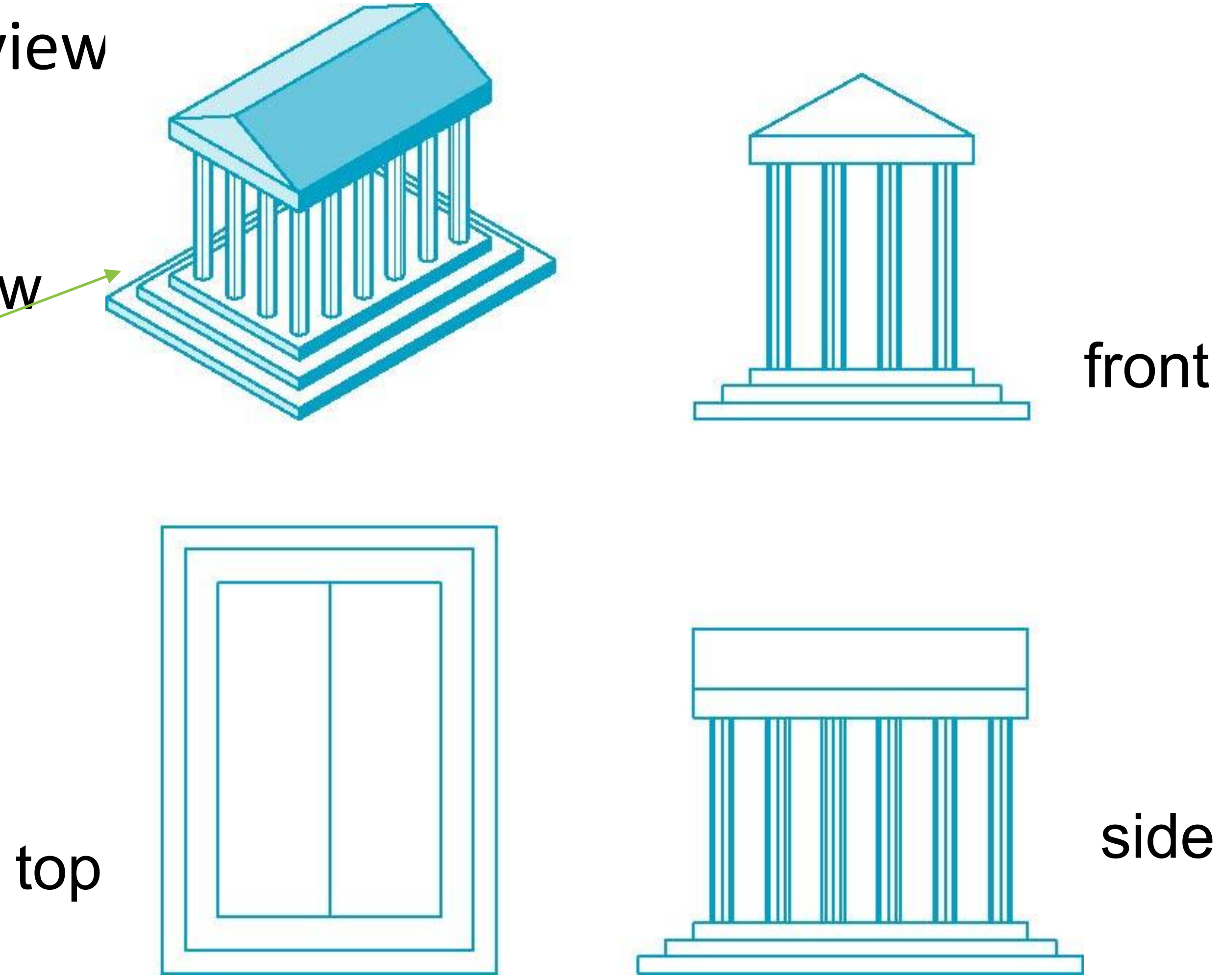
- DOP or all Projectors are orthogonal (perpendicular) to projection surface



Multiview Orthographic Projection

- Projection plane parallel to principal plane
- Usually form front, top, side view

isometric (not multiview
orthographic view)



In CAD and architecture, we often display three multiviews plus isometric

Multiview Orthographic projection

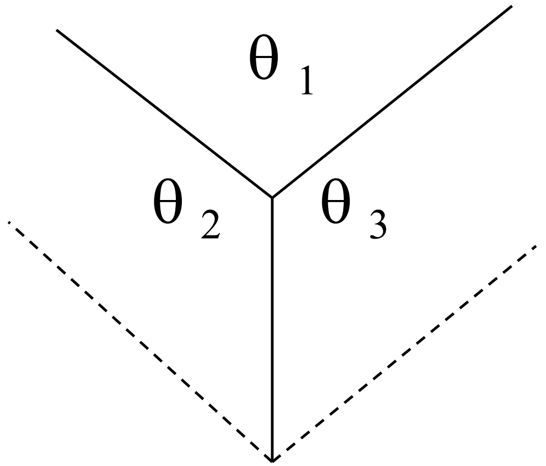
- Orthographic (or orthogonal) projections:
  - front elevation, top-elevation and side-elevation.
  - all have projection plane perpendicular to a principle axes.

- Useful because angle and distance measurements can be made...

- However, As only one face of an object is shown, it can be hard to create a mental image of the object, even when several view are available

Axonometric projection

- A type of parallel projection
  - Uses projection planes that are not normal to any principal axis.
- On the basis of *projection plane normal* ***N = (dx, dy, dz)*** subclasses are:

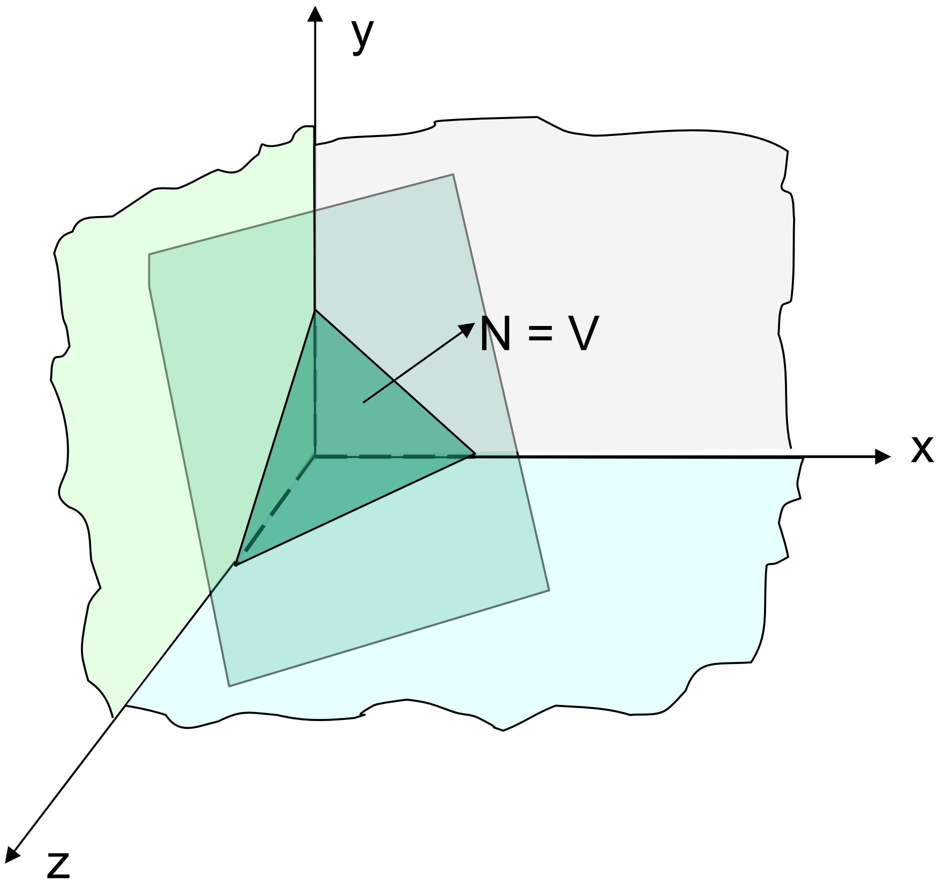


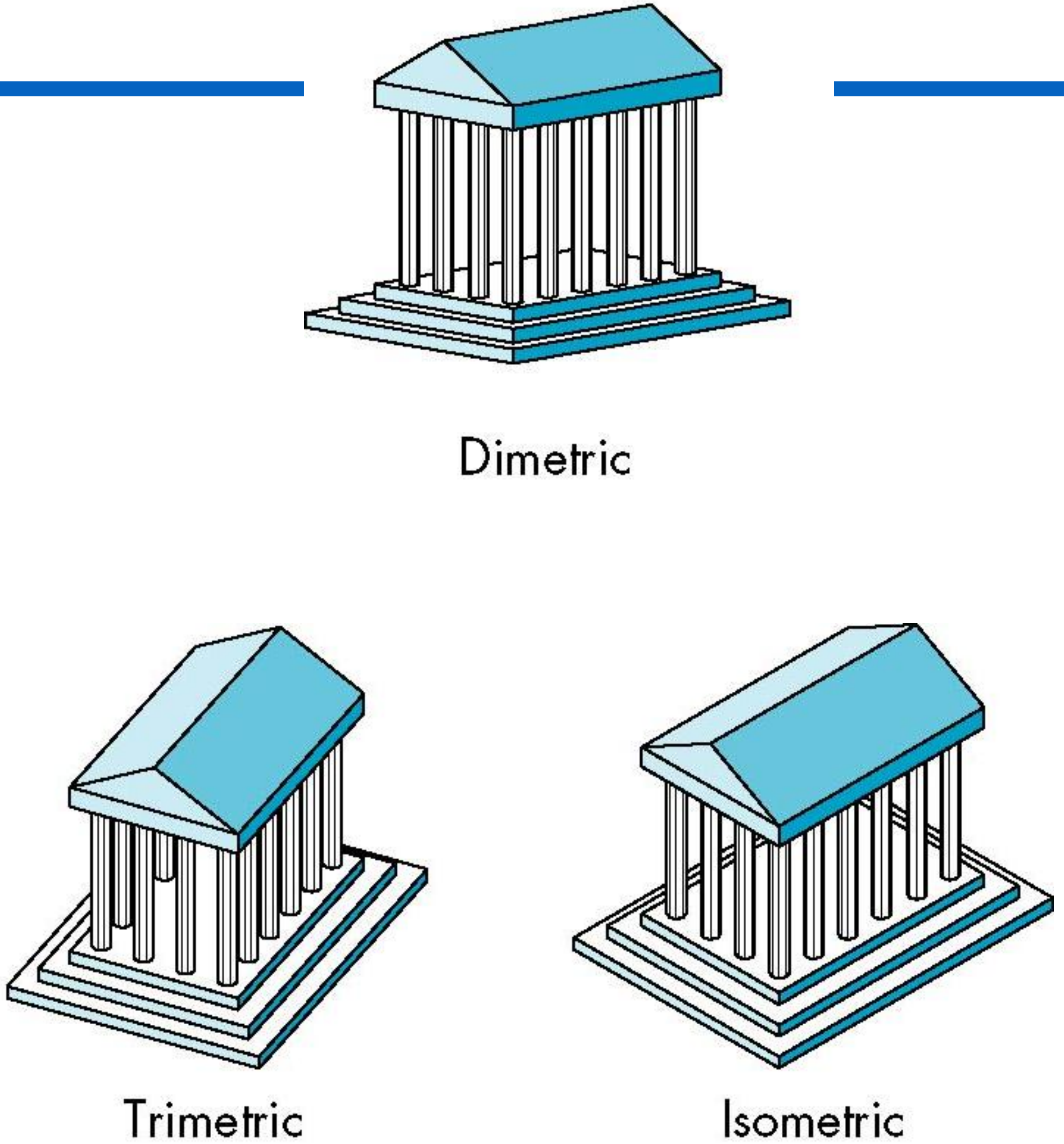
- ***Isometric*** : | ***dx*** **|** = **|** ***dy*** **|** = **|** ***dz*** **|** i.e. ***N*** makes equal angles with all principal axes.
- ***Dimetric*** : | ***dx*** **|** = **|** ***dy*** **|**
- ***Trimetric*** : | **dx |** **!=** **| dy |** **!=** **| dz |**

classify by how many angles of a corner of a projected cube are the same

none: trimetric
two: dimetric
three: isometric

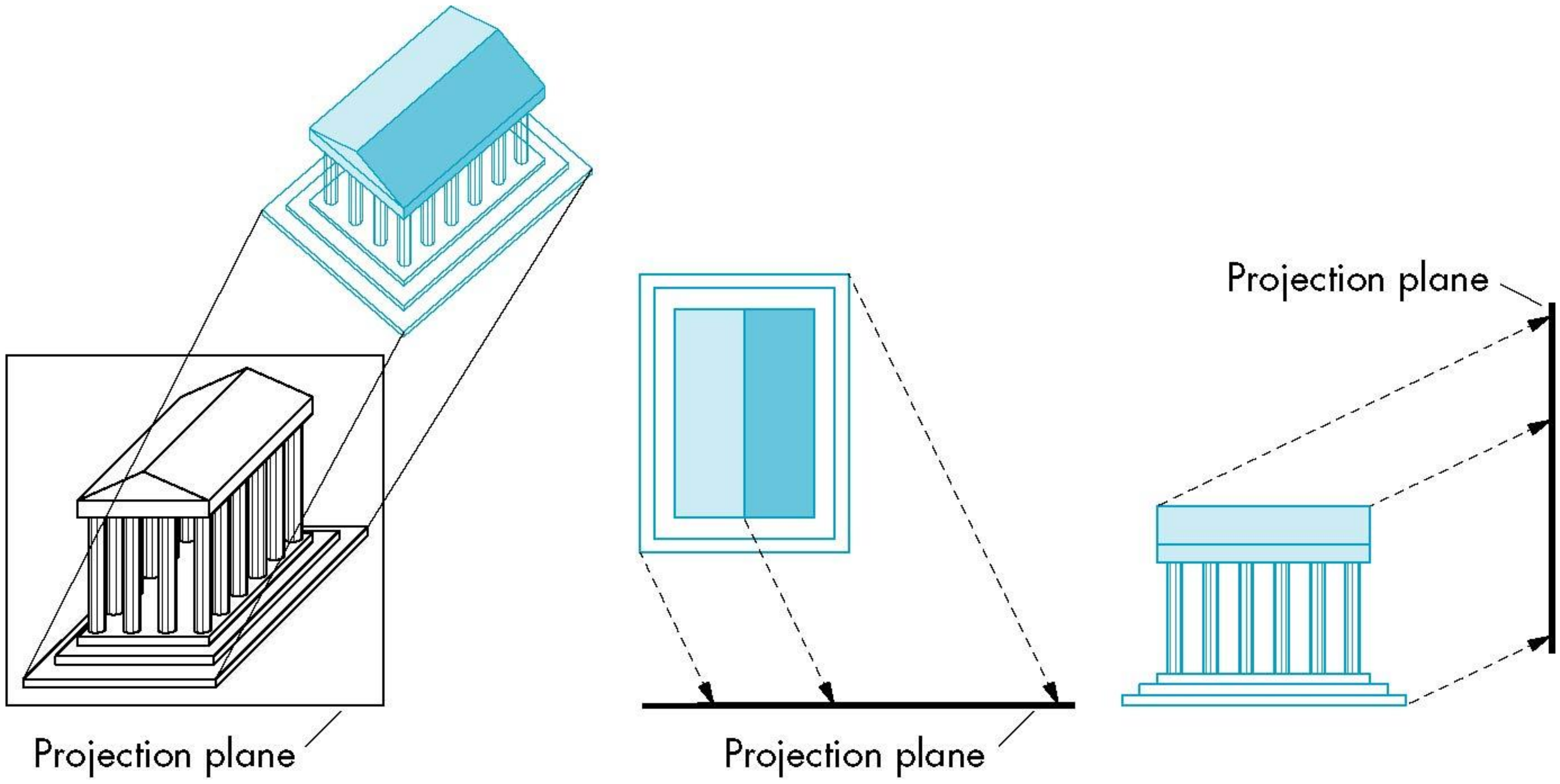
Axonometric projection





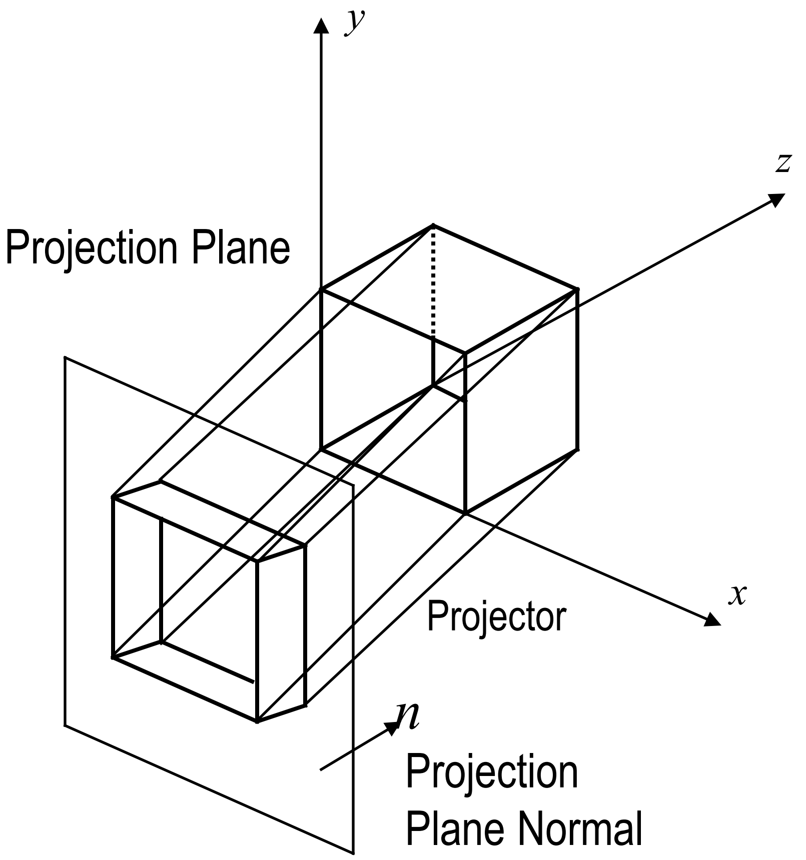
Oblique Projection

- Arbitrary relationship between projectors and projection plane



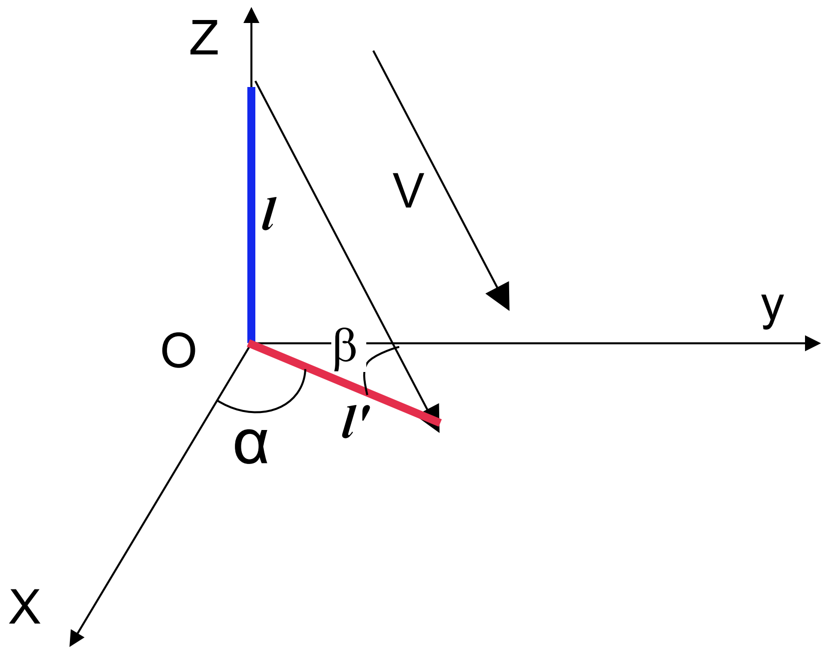
Oblique parallel projection

- Objects can be visualized better than orthographic projections
- Can measure distances, but not angles
  - Can only measure angles for faces of objects parallel to the plane
- Two types
  - ***Cavalier*** and ***Cabinet***



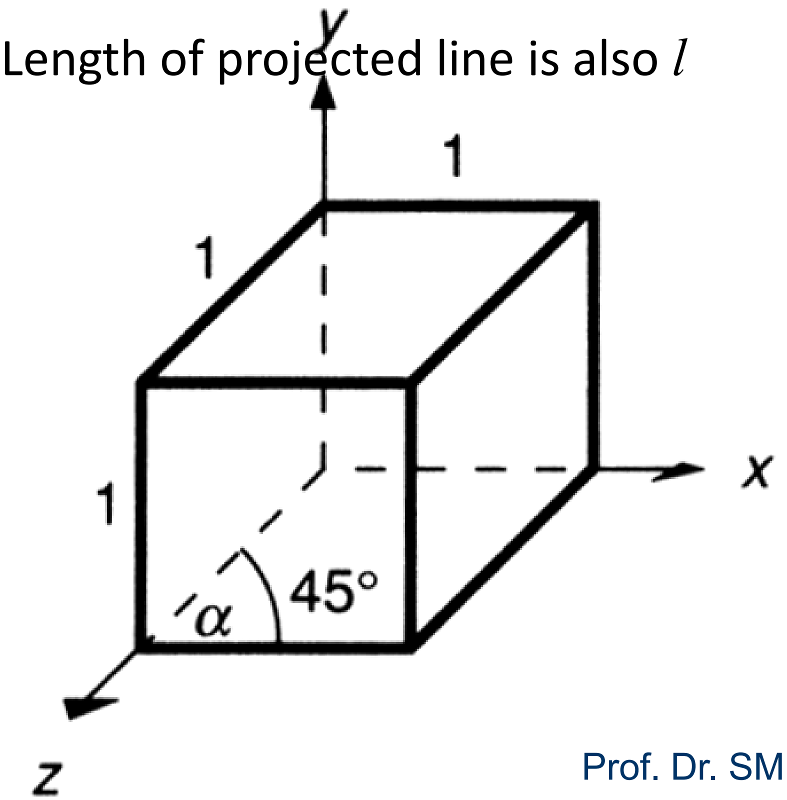
Oblique Projections

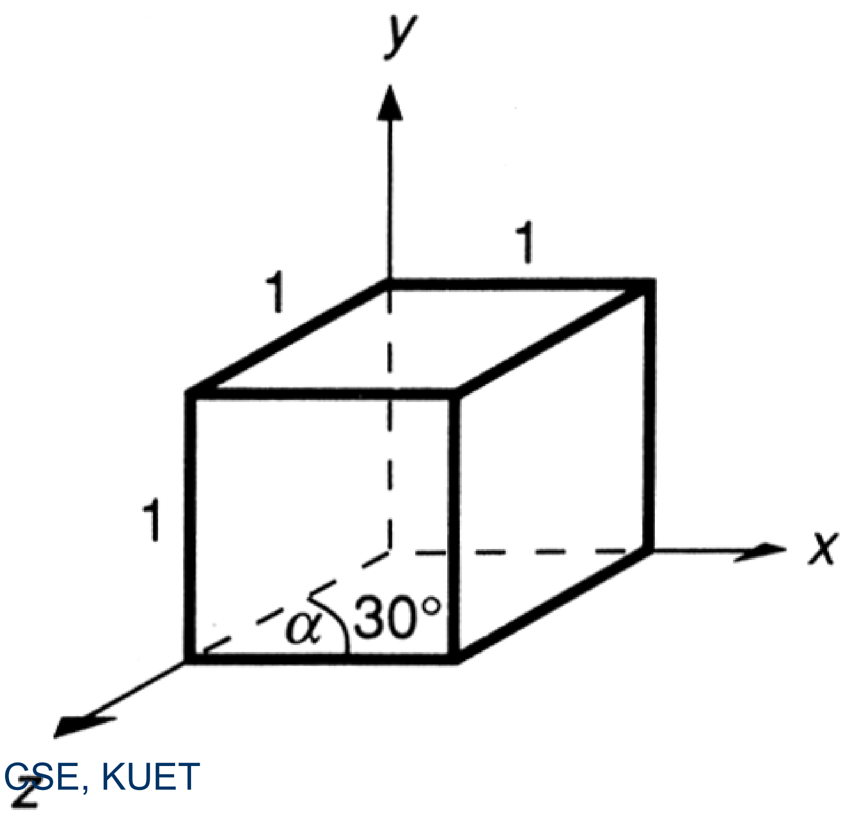
- Azimuth, α : is the angle the projection makes with x-axis
- Elevation, β : angle between view plane and direction of projection
- l : original length of a line perpendicular to view plane
- l': projected length of a line perpendicular to view plane



Oblique parallel projection

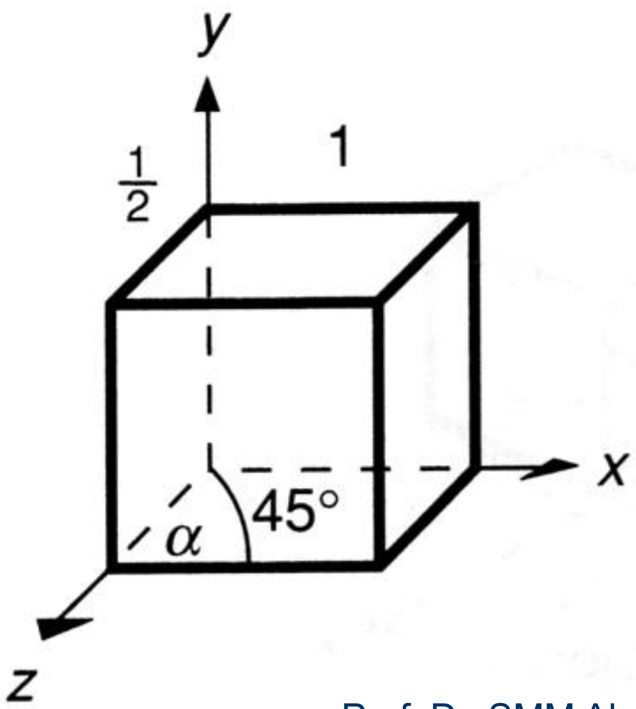
- Cavalier:
  - $l' = l$; β = 45°
  - The DOP makes a 45 degree angle with the projection plane.
  - There is no foreshortening
    - Length of any line perpendicular to view plane is l
    - The Length of projected line is also l

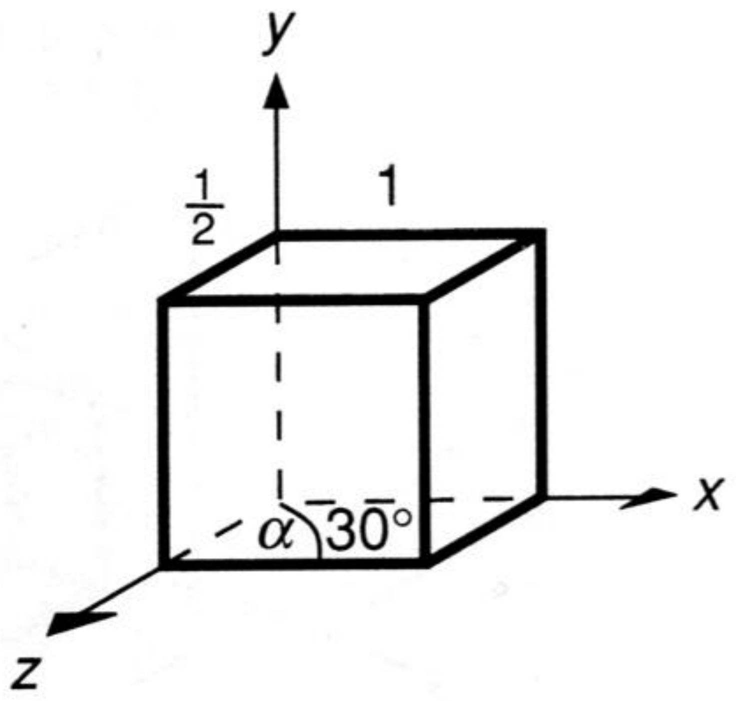




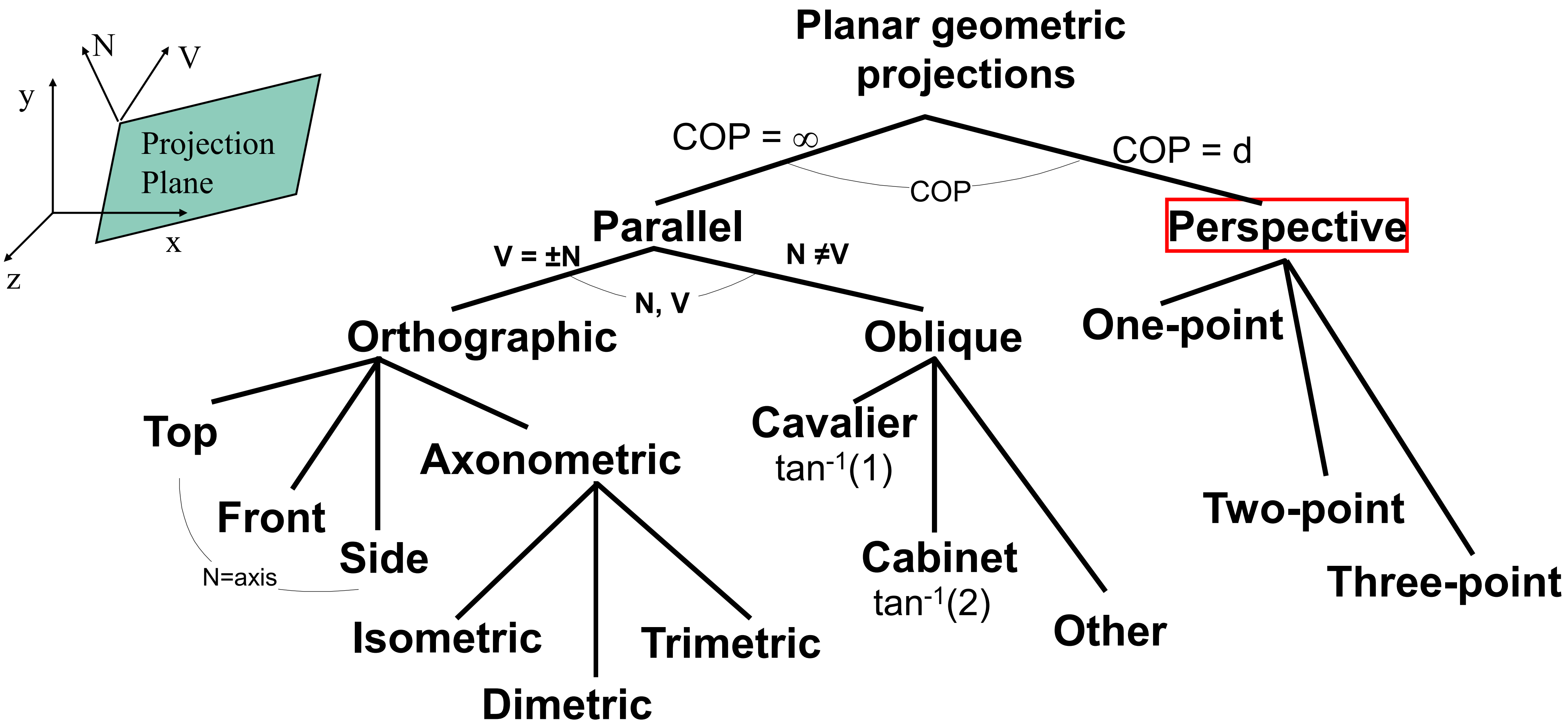
Oblique parallel projection

- Cabinet:
  - $l' = l/2$; β = 63.4°
  - The DOP makes a 63.4 degree angle with the projection plane.
  - This results in foreshortening of the z axis, and provides a more “realistic” view





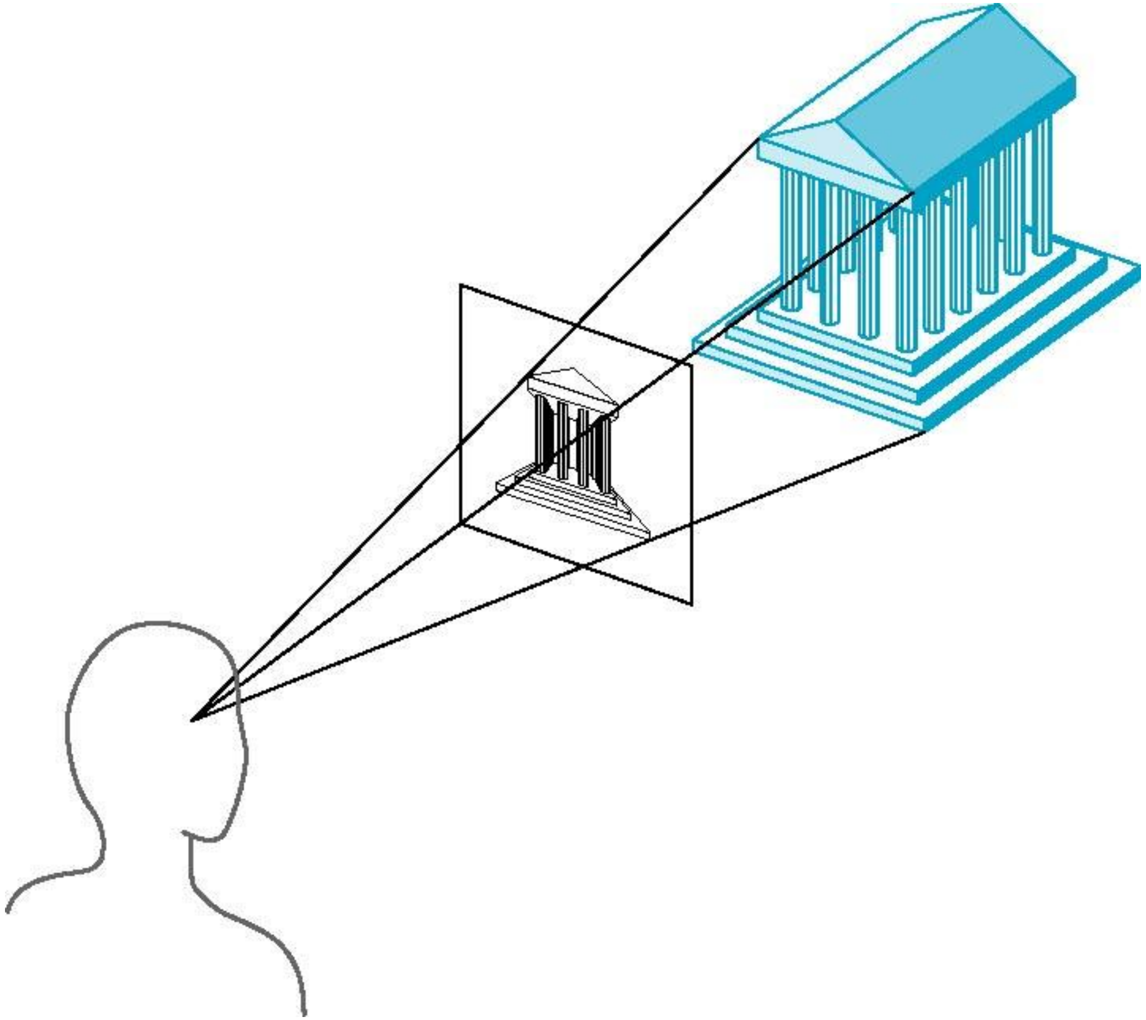
Planner Geometric Projections taxonomy



Subclasses of planar geometric projections

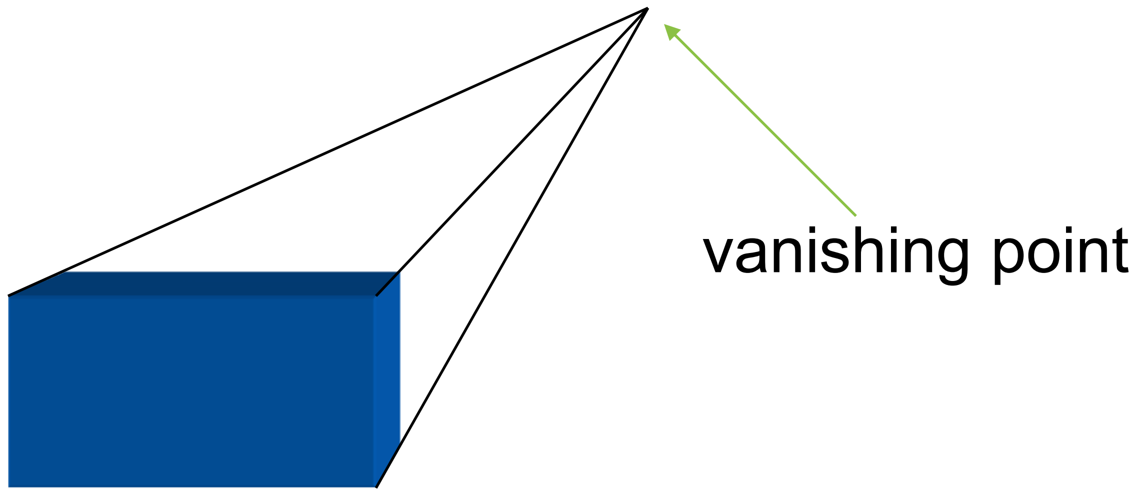
Perspective Projection

- Map points onto "view plane" along "projectors" emanating from "center of projection"(cop)
- Projectors converge at COP



Vanishing Points

- Parallel lines (not parallel to the projection plan) on the object converge at a single point in the projection (the *vanishing point*)
- Drawing simple perspectives by hand uses these vanishing point(s)

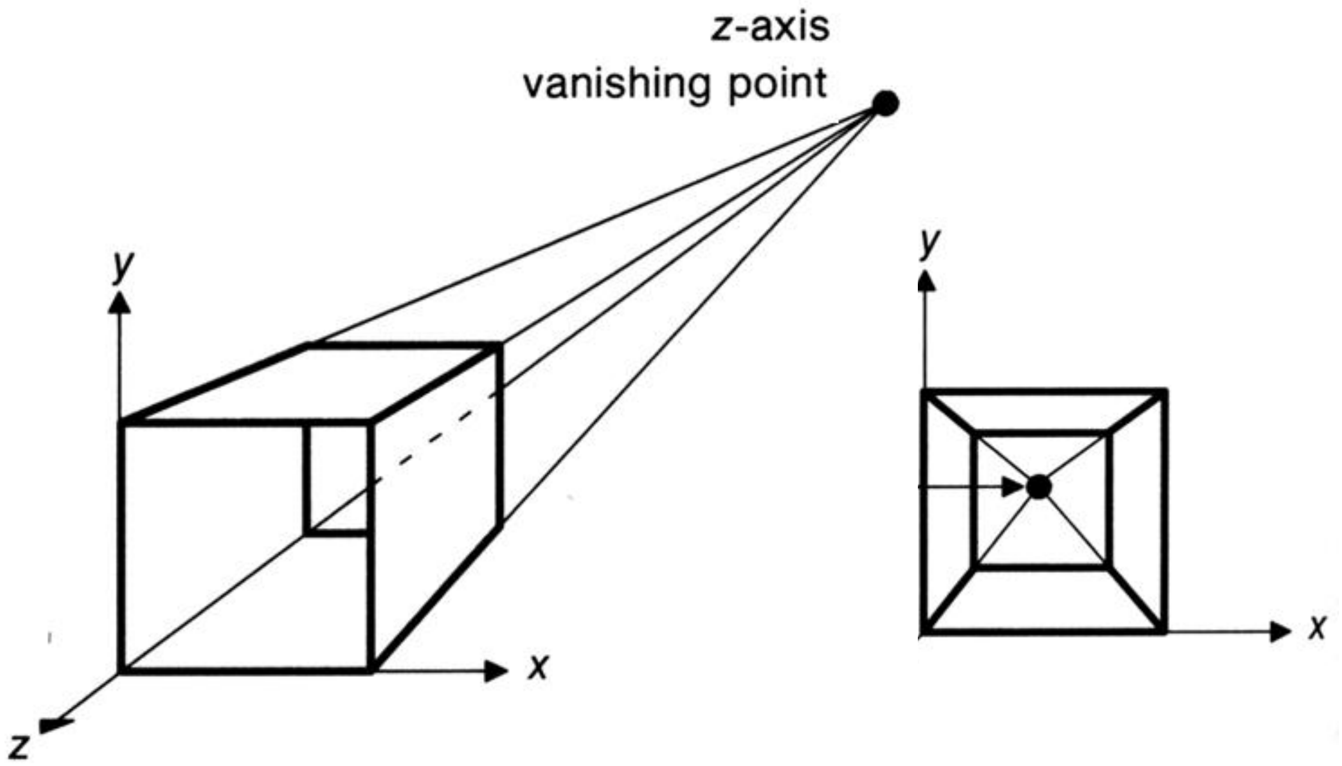


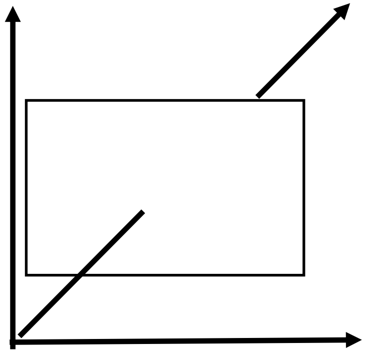
Perspective Projection - types

- Based on vanishing points
- If a set of lines are parallel to one of the three axes, the vanishing point is called an axis vanishing point (Principal Vanishing Point).
- There are at most 3 such points, corresponding to the number of axes cut by the projection plane
- One-point / two-point / three-point perspective:
  - One / two / three principle axis cut by projection plane

Perspective Projection

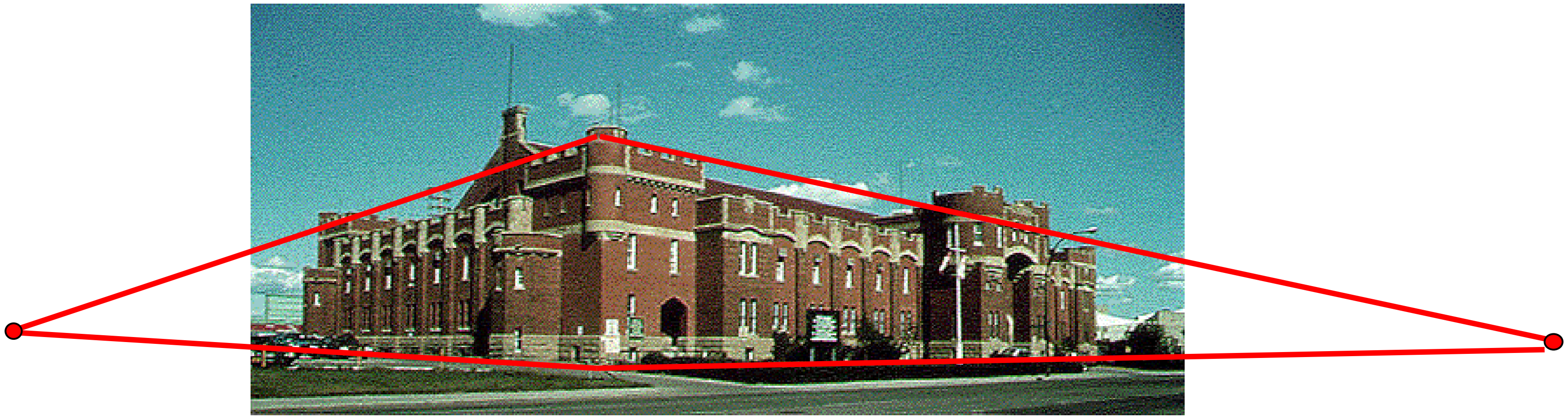
- One point perspective projection of a cube
  - X and Y parallel lines do not converge

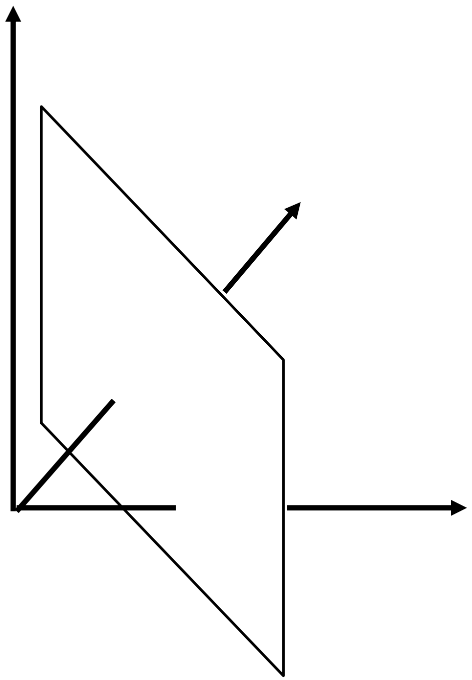




Perspective Projection

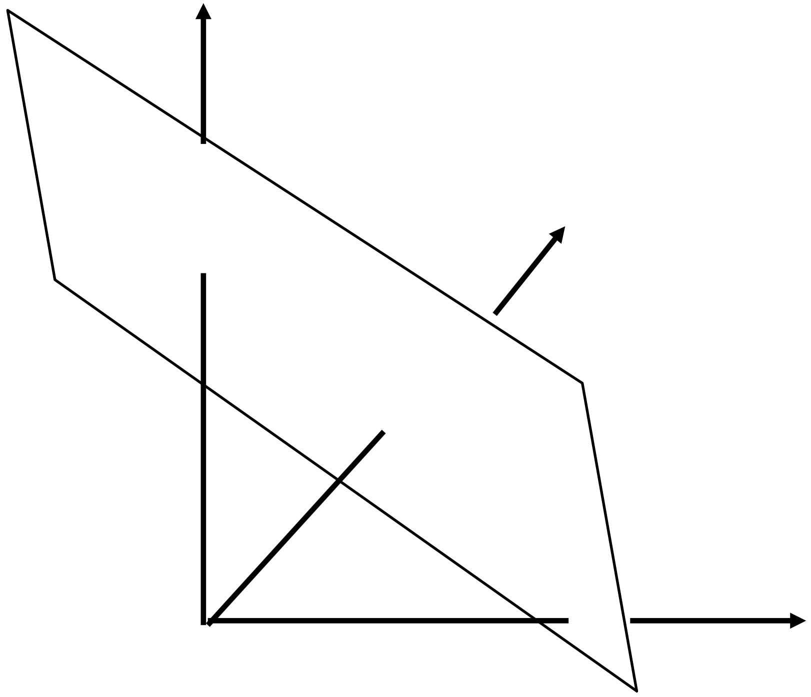
- Two-point perspective:
  - often used in architectural, engineering and industrial design drawings.

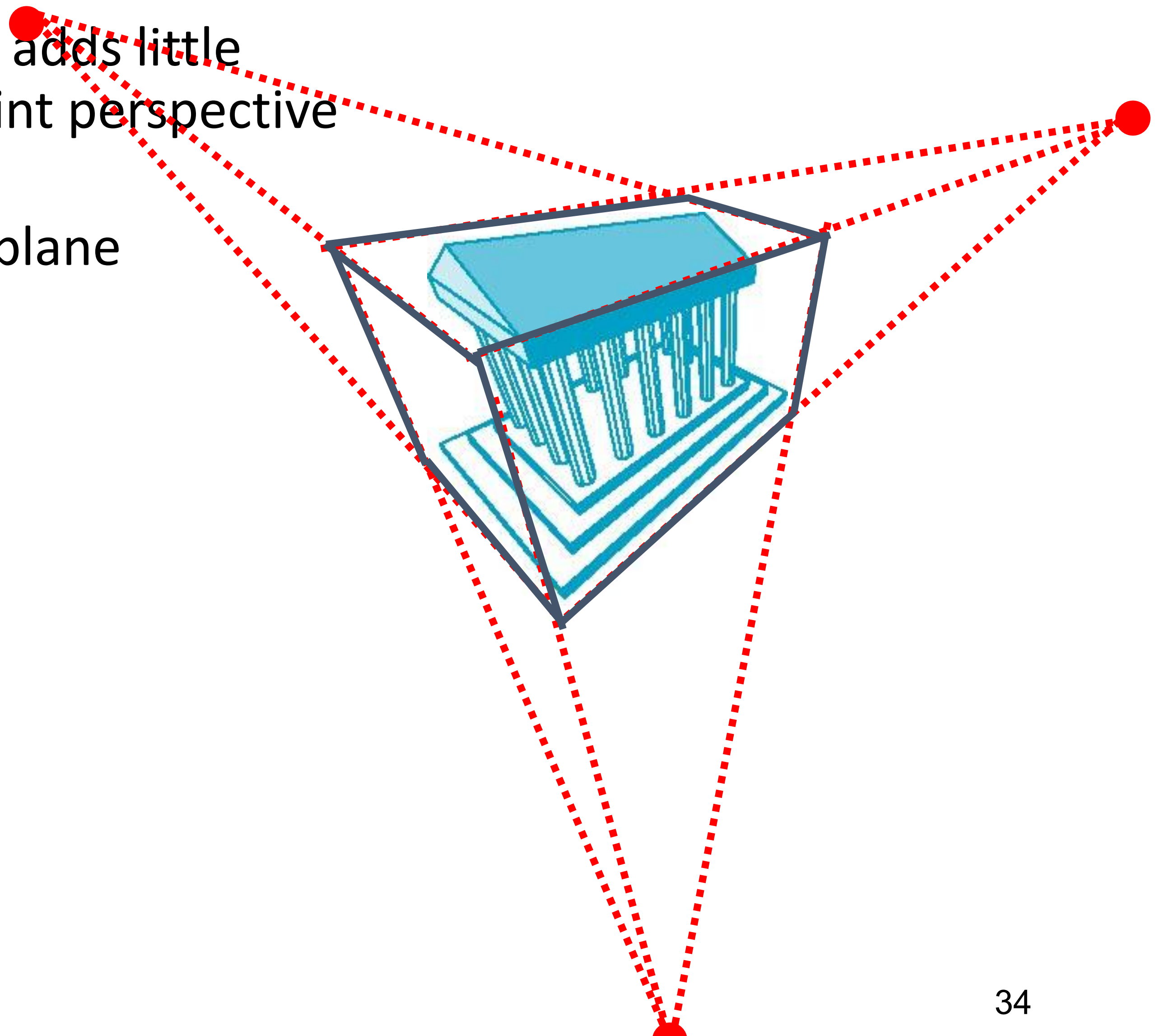




Perspective Projection

- Three-Point Perspective
  - Three-point is used less frequently as it adds little extra realism to that offered by two-point perspective projection.
  - No principal face parallel to projection plane

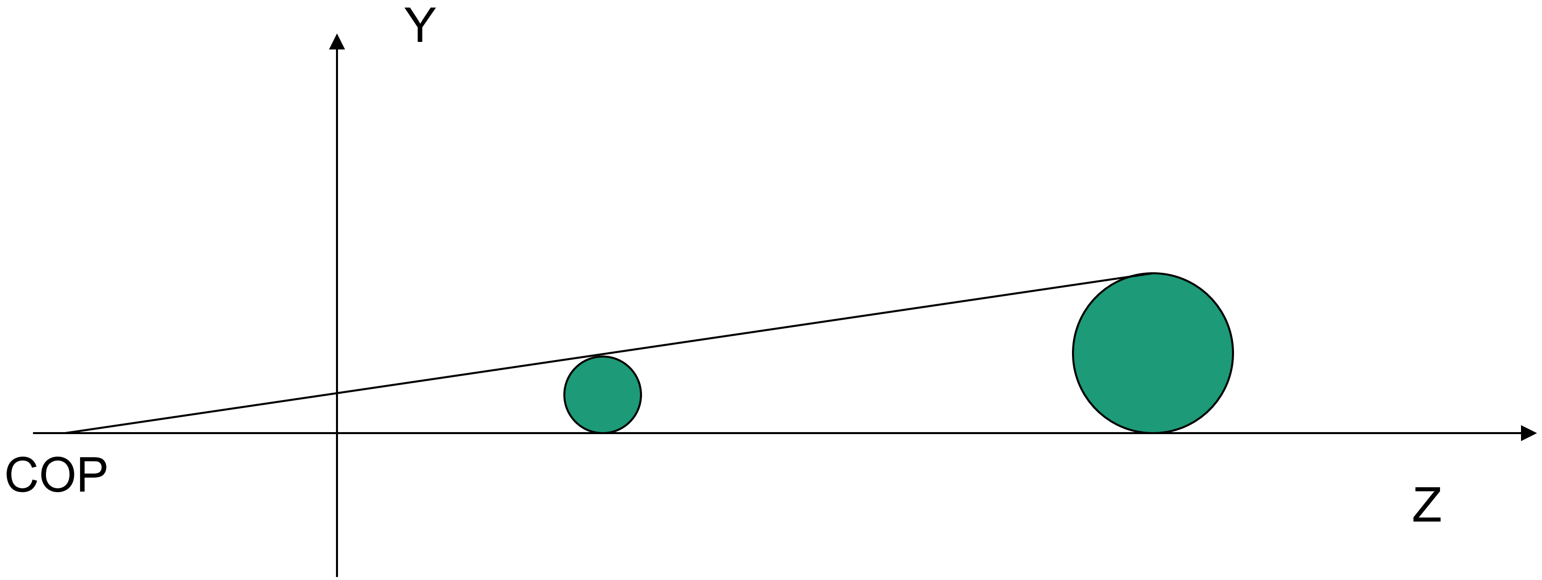




Perspective Projection-Anomalies

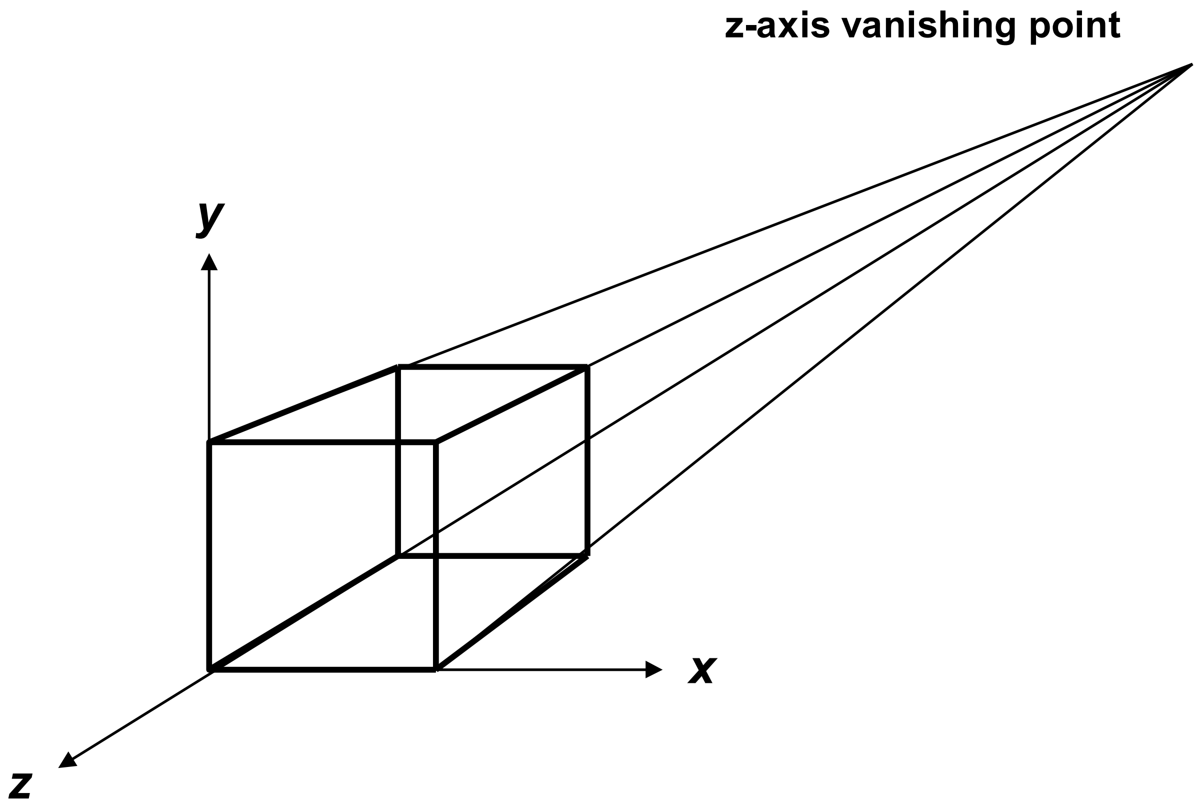
- Enhances realism in terms of depth cue but distorts sizes and shapes

- ***Perspective foreshortening*** The farther an object is from COP the smaller it appears



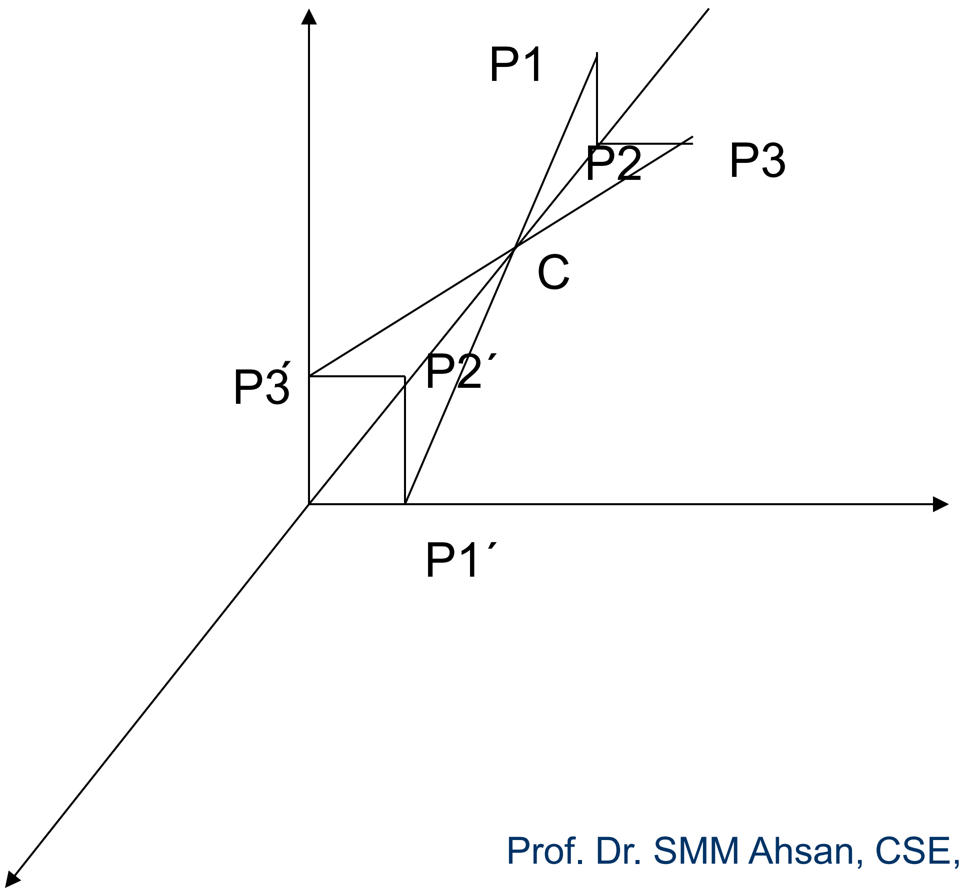
Perspective Projection-Anomalies

- ***Vanishing Points:*** Any set of parallel lines not parallel to view plane (or not perpendicular to view plane normal) appear to meet at some point.



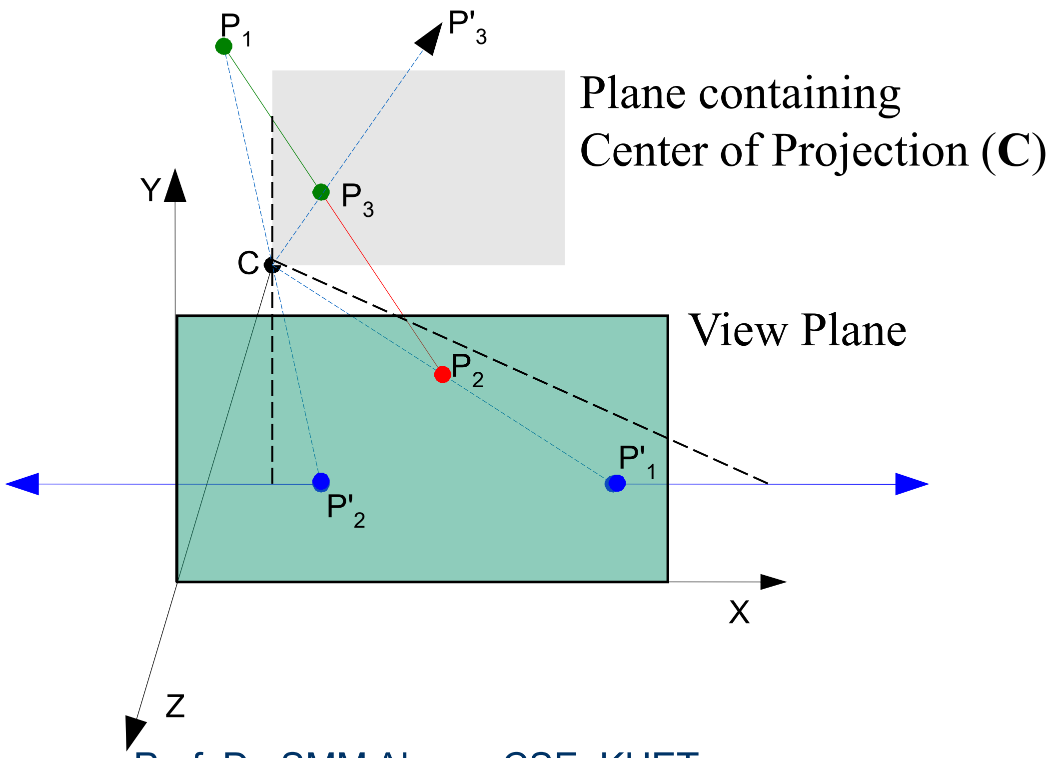
Perspective Projection-Anomalies

- ***View Confusion:*** Objects behind the center of projection are projected upside down and backward onto the view-plane.



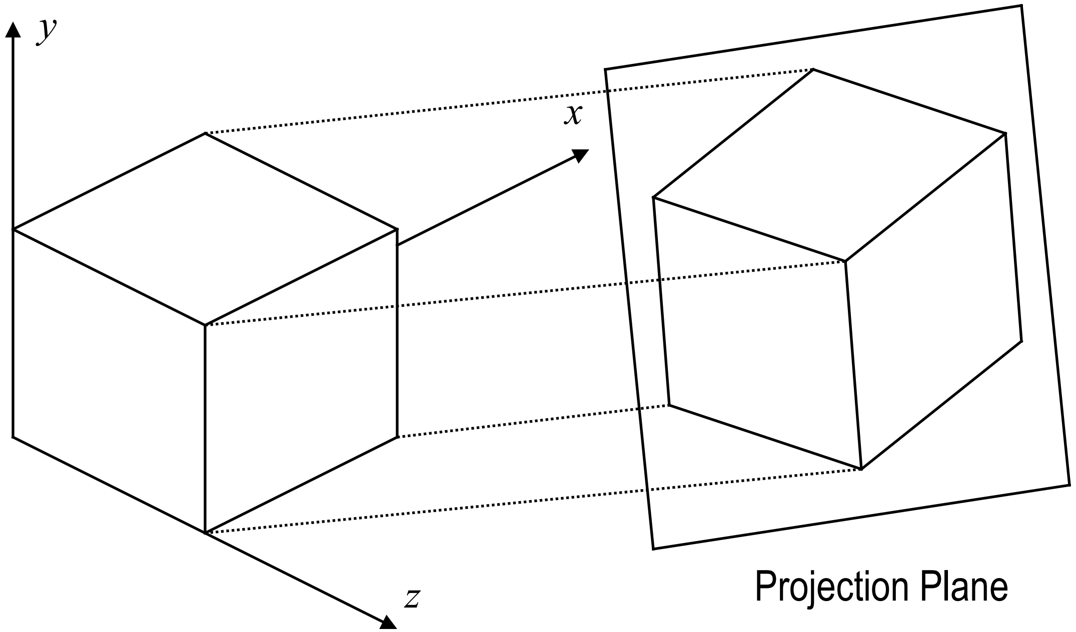
Perspective Projection-Anomalies

- ***Topological distortion:*** A line segment joining a point which lies in front of the viewer to a point in back of the viewer is projected to a broken line of infinite extent.



Axonometric vs Perspective

- Axonometric projection shows several faces of an object at once like perspective projection.
- But the foreshortening is uniform rather than being related to the distance from the COP.



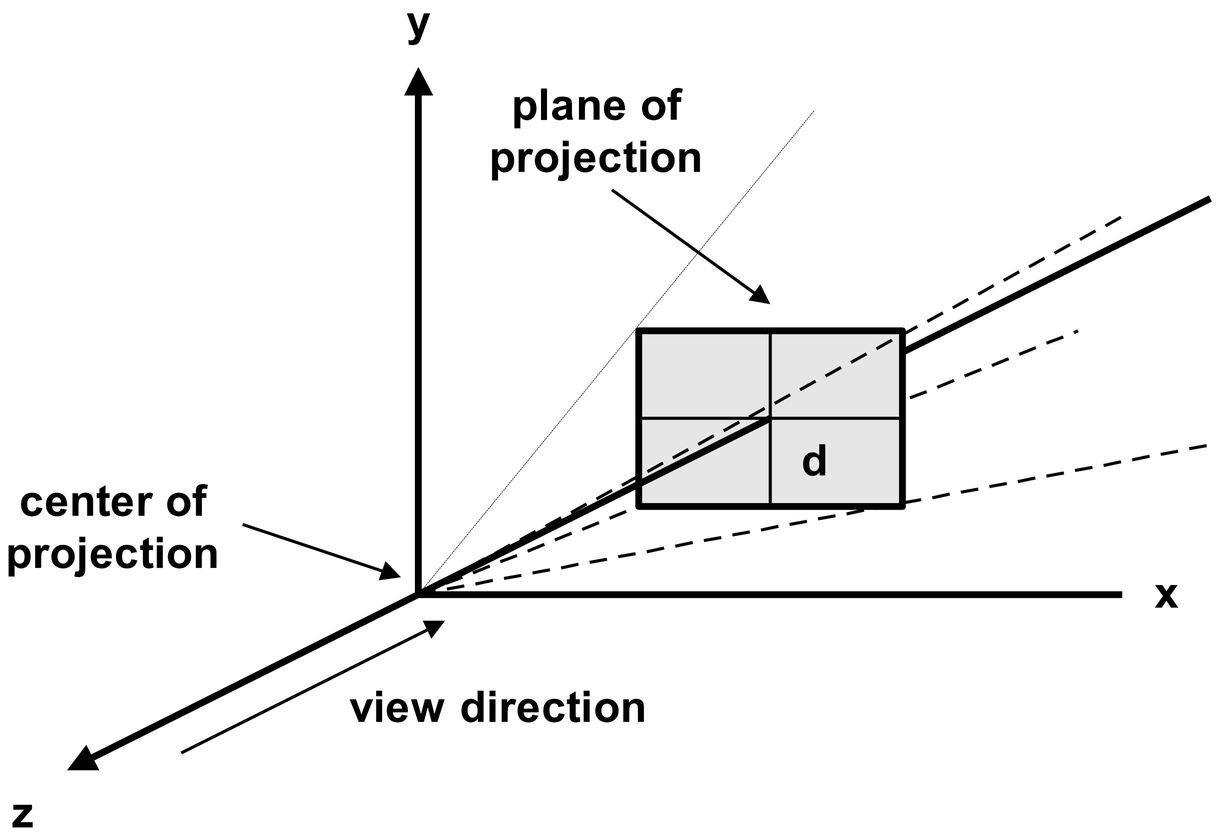
Isometric proj

Projection Mathematics

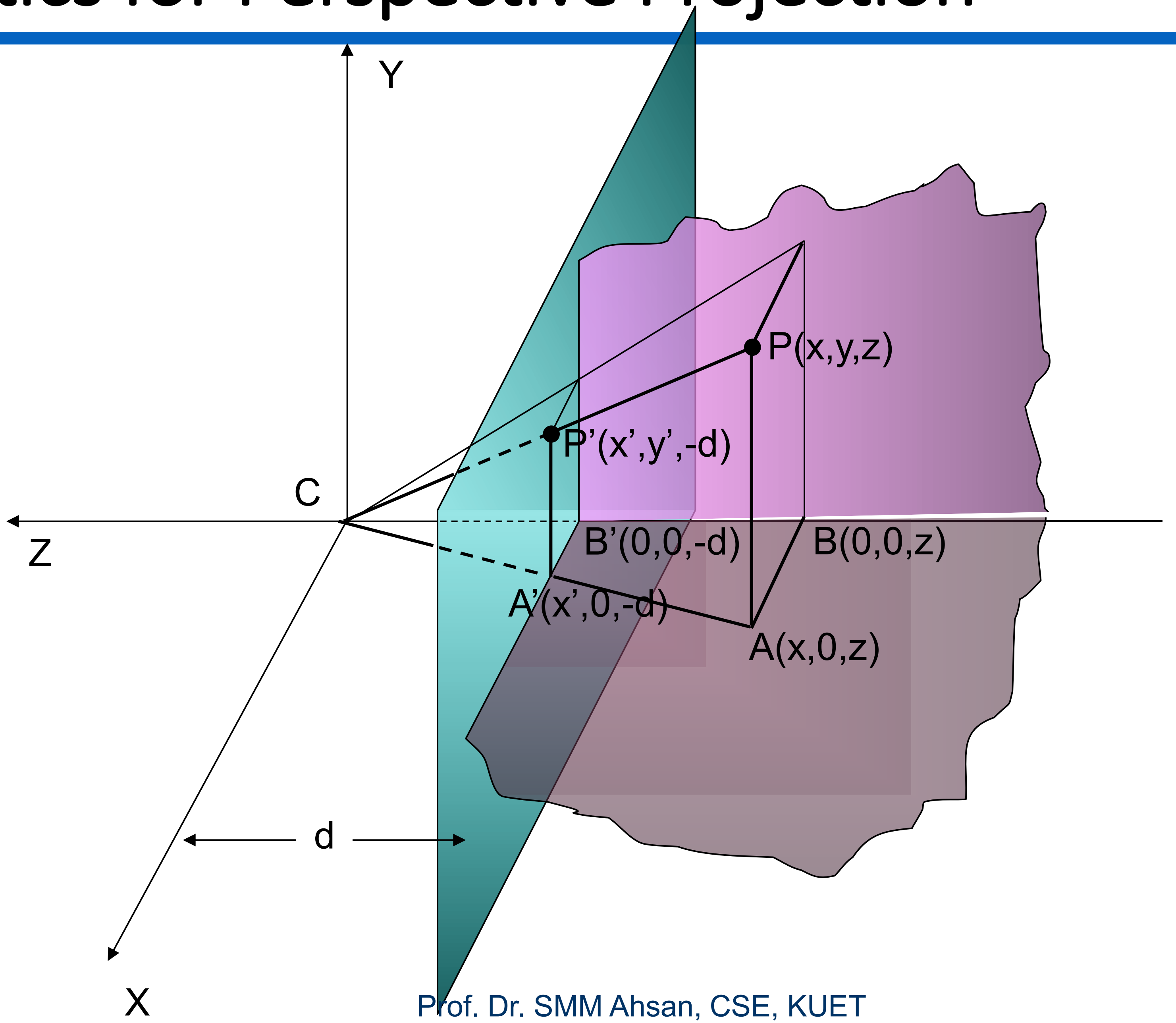
Projective Transformations

- Projection plane, COP, etc. all are defined in VCS

Settings for perspective projection



Mathematics for Perspective Projection



Mathematics for Perspective Projection

- From triangle ABC and A'B'C

$$\frac{AB}{BC} = \frac{A'B'}{B'C}$$

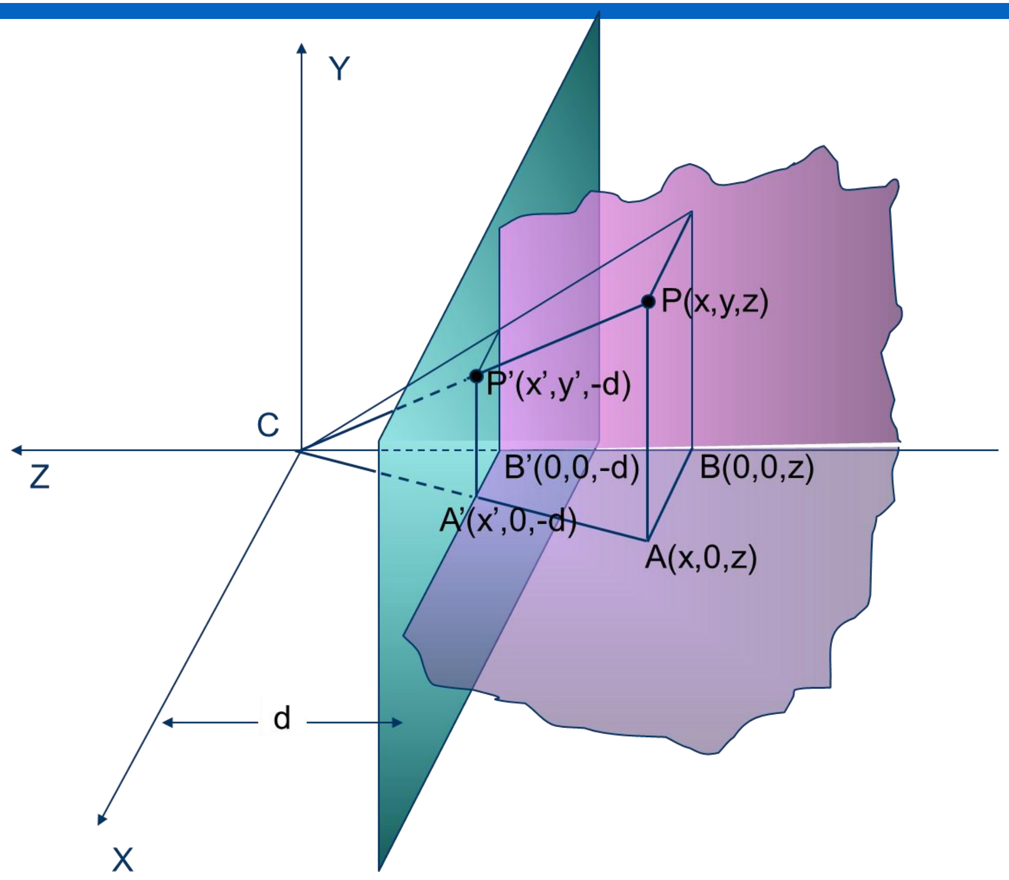
$$\frac{x}{z} = \frac{x'}{-d} \Rightarrow x' = \frac{x}{-(z/d)}$$

similarly, $y' = \dfrac{y}{-(z/d)}$ and,

$$z' = -d$$

$$(x', y', z', 1) \Rightarrow \left(\frac{x}{-(z/d)}, \frac{y}{-(z/d)}, -d, 1 \right) \equiv (x, y, z, -(z/d))$$

Some text book doesn't use the minus sign here, both are OK. So, be careful which convention you are using



Matrices for Projective Trans.

$$\begin{pmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{x}{-(z/d)} \\ \dfrac{y}{-(z/d)} \\ -d \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ -\dfrac{z}{d} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\dfrac{1}{d} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ -\dfrac{z}{d} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \\ -\dfrac{z}{d} \end{pmatrix} \xrightarrow{\text{perspective division}} \begin{pmatrix} \dfrac{x}{-(z/d)} \\ \dfrac{y}{-(z/d)} \\ -d \\ 1 \end{pmatrix}$$

Matrices for Projective Trans.

Projection plane cuts the x axis

$$\mathbf{M}_{PER} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{d_x} & 0 & 0 & 1 \end{bmatrix}$$

Projection plane cuts the y axis

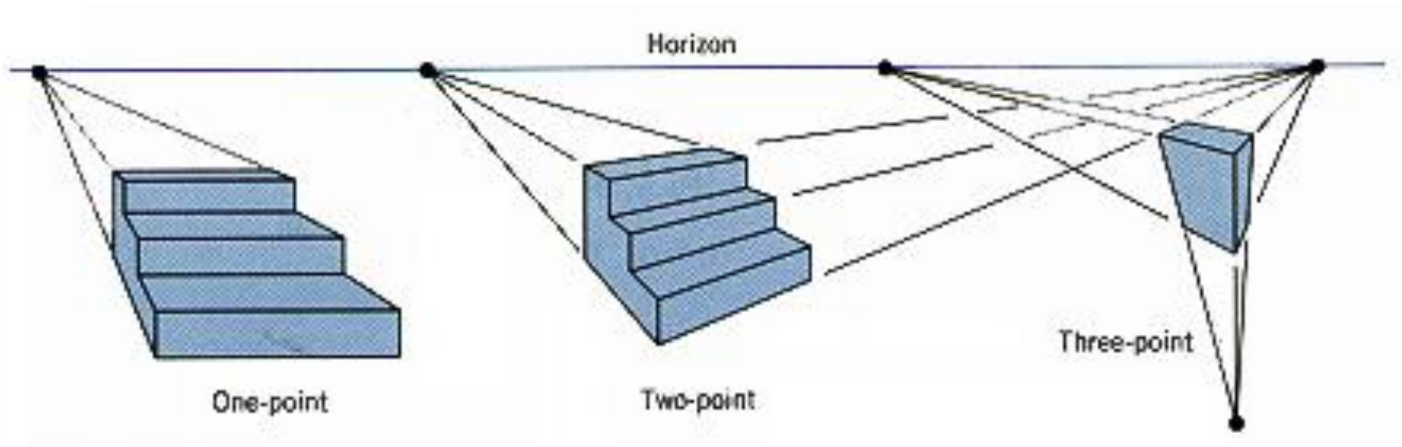
$$\mathbf{M}_{PER} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{d_y} & 0 & 1 \end{bmatrix}$$

Perspective Projection Matrix

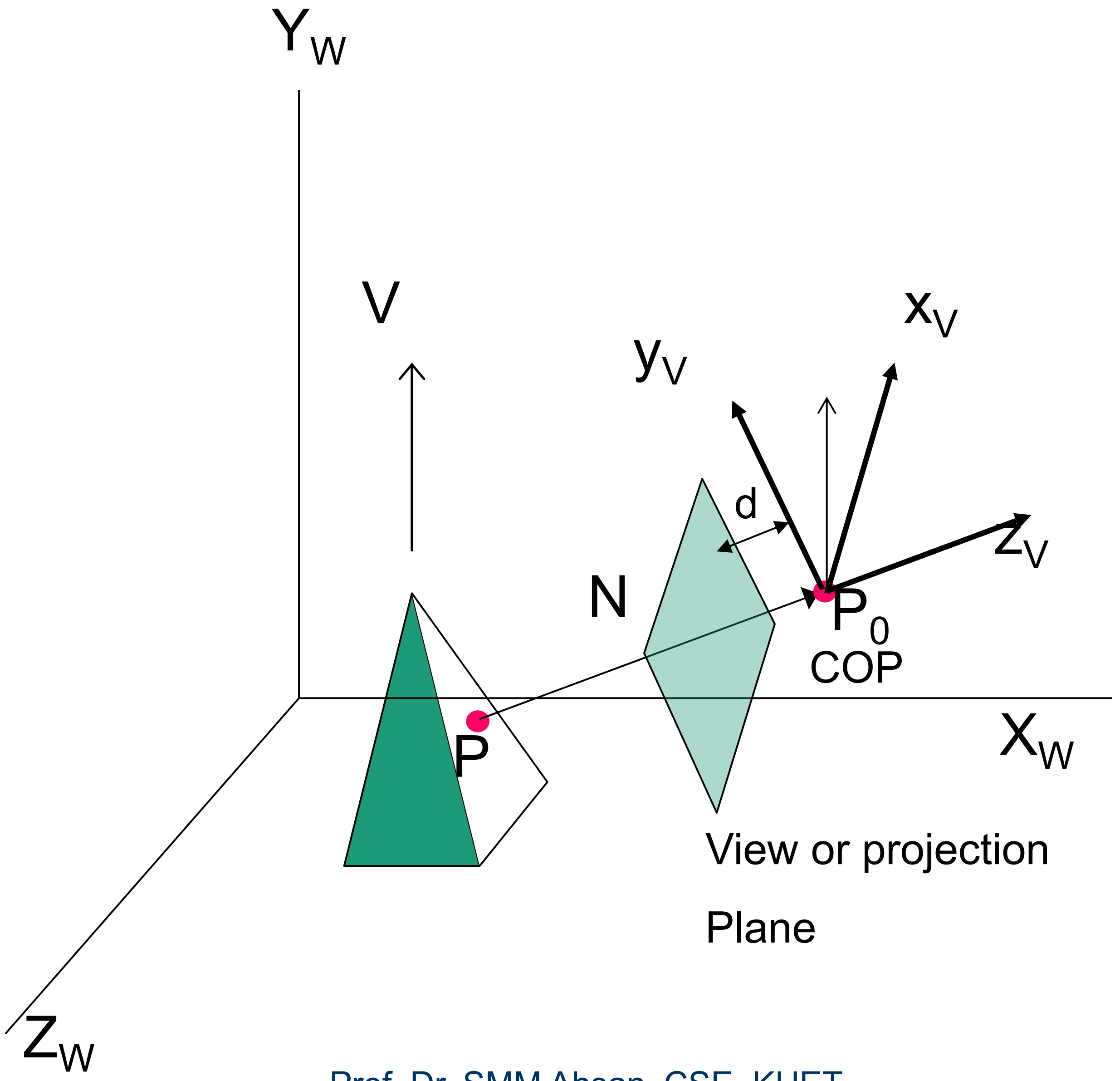
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ r & s & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & s & t & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ r & s & t & 1 \end{bmatrix}$$

2-point perspectives

3-point perspectives

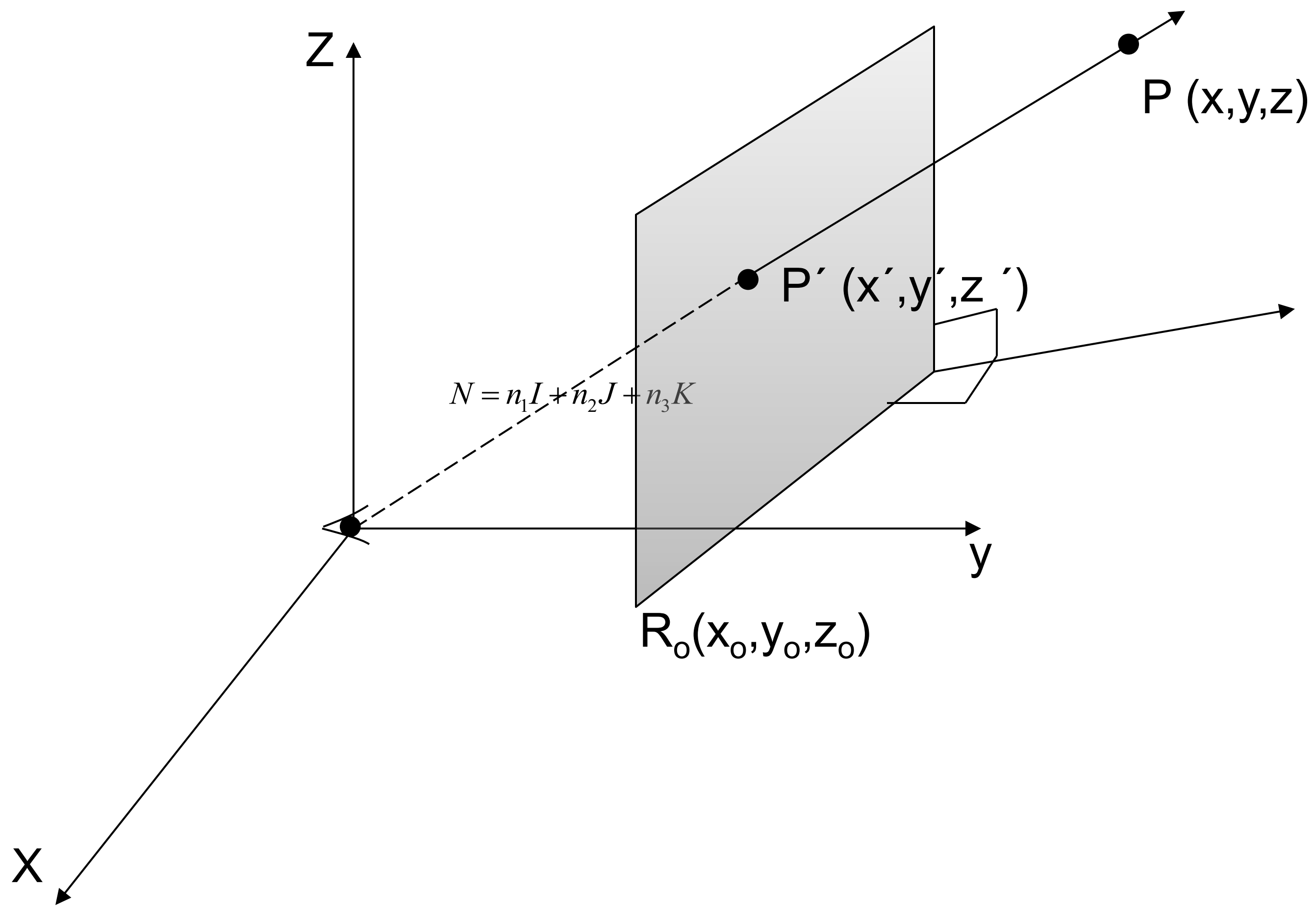


Rel^n with VCS and projection



Perspective Projection – arbitrary plane

Using origin as COP, projection plane is a plane with normal N passing through point R_0



Perspective Projection – arbitrary plane

$\overline{OP}$ and $\overline{OP'}$ are same direction

$$\overline{OP'} = \alpha \overline{OP}$$

$$x' = \alpha x, \quad y' = \alpha y \quad z' = \alpha z$$

$$N.\overline{P'R_0} = 0$$

$$n_1(x' - x_0) + n_2(y' - y_0) + n_3(z' - z_0) = 0$$

$$n_1 x' + n_2 y' + n_3 z' = d_0 \quad , \text{ where } d_0 = n_1 x_0 + n_2 y_0 + n_3 z_0$$

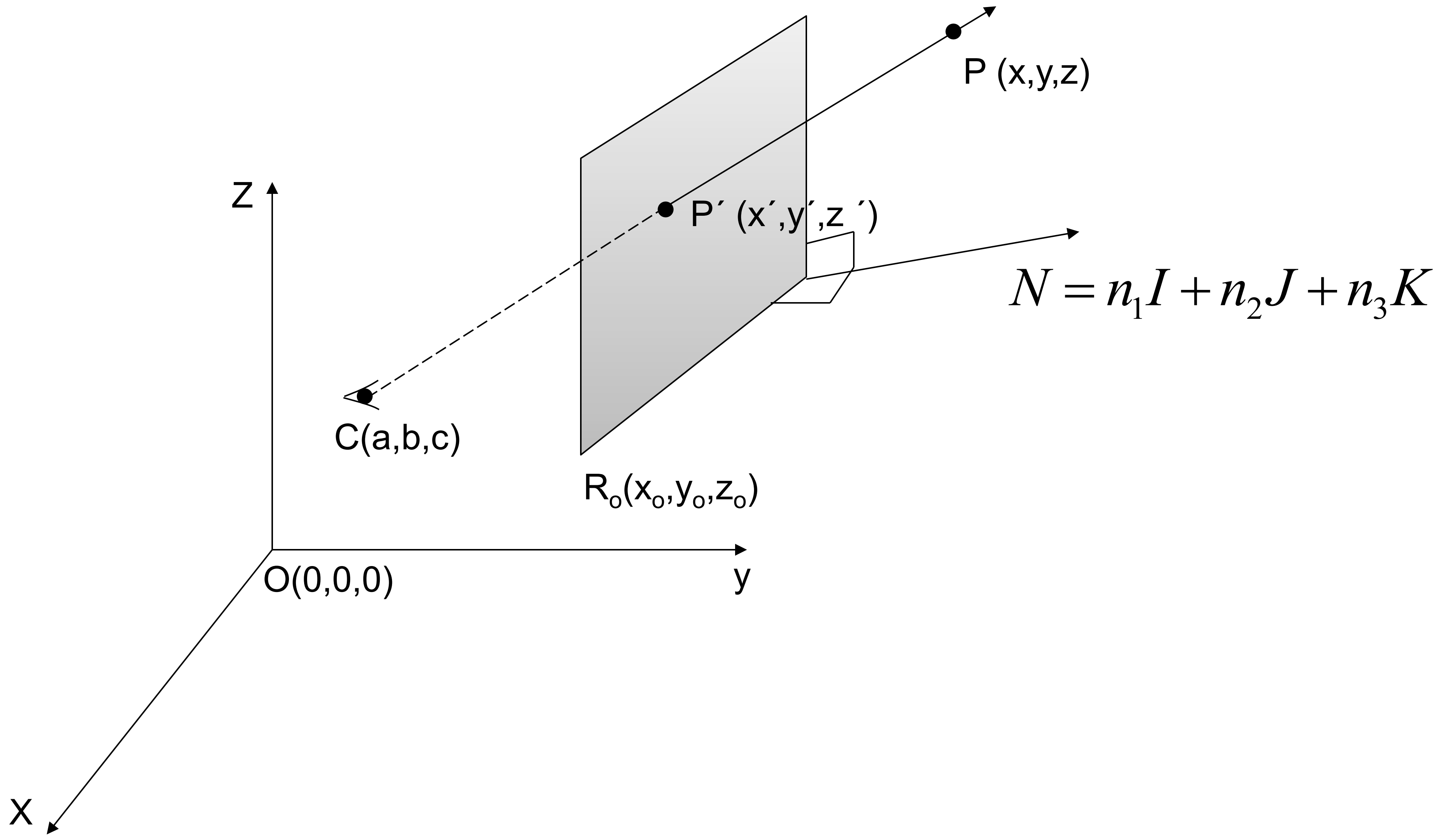
$$\alpha = \frac{d_0}{n_1 x + n_2 y + n_3 z}$$

Perspective Projection – arbitrary plane

$$P' \quad = \quad Per_{N,R_0} \quad . \quad P$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} d_0.x \\ d_0.y \\ d_0.z \\ n_1x + n_2y + n_3z \end{bmatrix} = \begin{bmatrix} d_0 & 0 & 0 & 0 \\ 0 & d_0 & 0 & 0 \\ 0 & 0 & d_0 & 0 \\ n_1 & n_2 & n_3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Perspective Projection



Perspective Projection

$P'C = \alpha\ PC$

$x' = \alpha(x-a) + a; \quad y' = \alpha(y-b) + b; \quad z' = \alpha(z-c) + c$

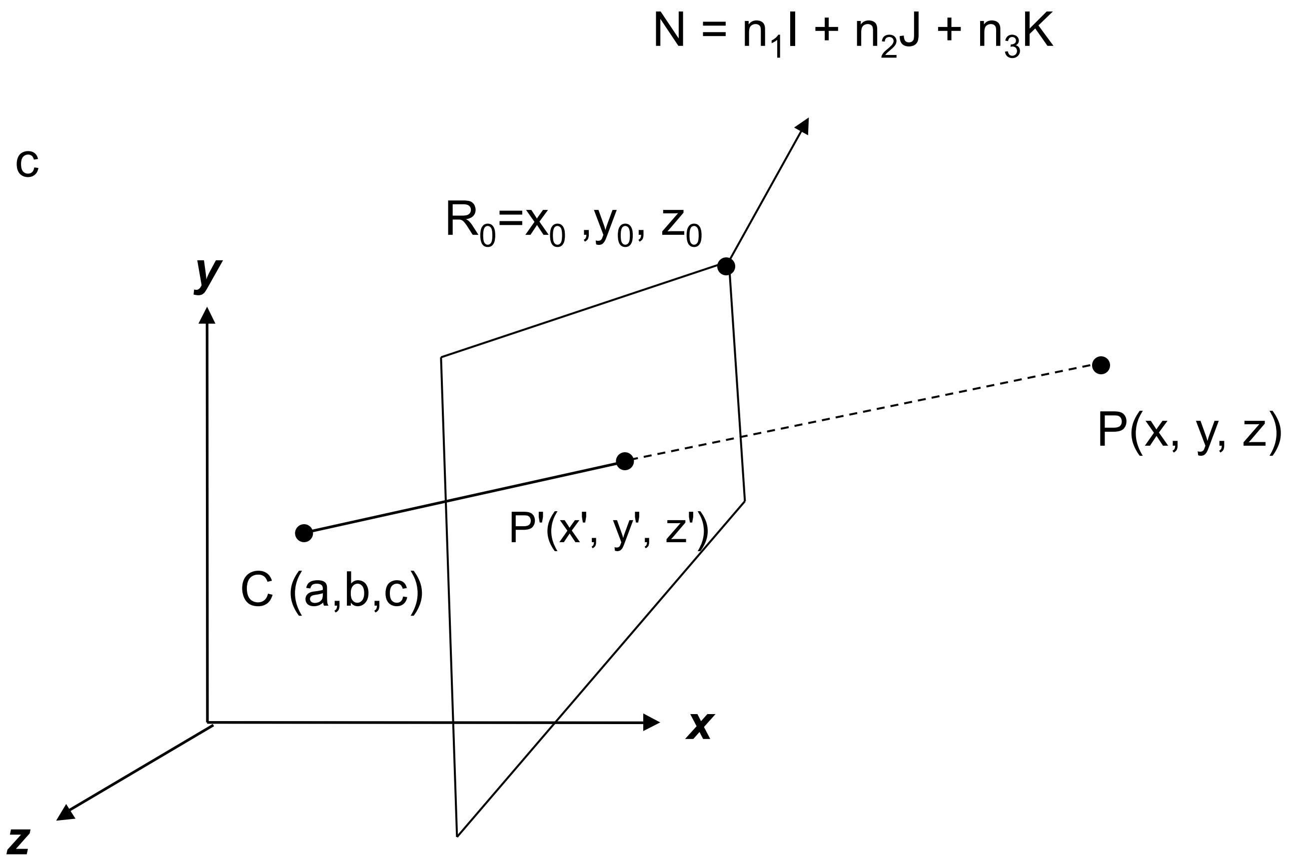
$n_1(x' - x_0) + n_2(y' - y_0) + n_3(z' - z_0) = 0$

$n_1x' + n_2y' + n_3z' = d_0$

$$\alpha = \frac{d}{n_1(x-a)+n_2(y-b)+n_3(z-c)}$$

$d = (n_1x_0 + n_2y_0 + n_3z_0) – (n_1a + n_2b + n_3c)$

$= d_0 – d_1$



Perspective Projection - DIY

- Follow the steps –
  - Translate so that C lies at the origin, hence, $R_0=(x_0 - a, y_0 - b, z_0 - c)$
  - Per
  - Translate back

$$\begin{bmatrix}1 & 0 & 0 & a\\0 & 1 & 0 & b\\0 & 0 & 1 & c\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}d & 0 & 0 & 0\\0 & d & 0 & 0\\0 & 0 & d & 0\\n_1 & n_2 & n_3 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & -a\\0 & 1 & 0 & -b\\0 & 0 & 1 & -c\\0 & 0 & 0 & 1\end{bmatrix}$$

$$\begin{pmatrix}d+an_1 & an_2 & an_3 & -ad_0\\bn_1 & d+bn_2 & bn_3 & -bd_0\\cn_1 & cn_2 & d+cn_3 & -cd_0\\n_1 & n_2 & n_3 & -d_1\end{pmatrix}$$

$d = (n_1x_0 + n_2y_0 + n_3z_0) – (n_1a + n_2b + n_3c)$

$= d_0 – d_1$

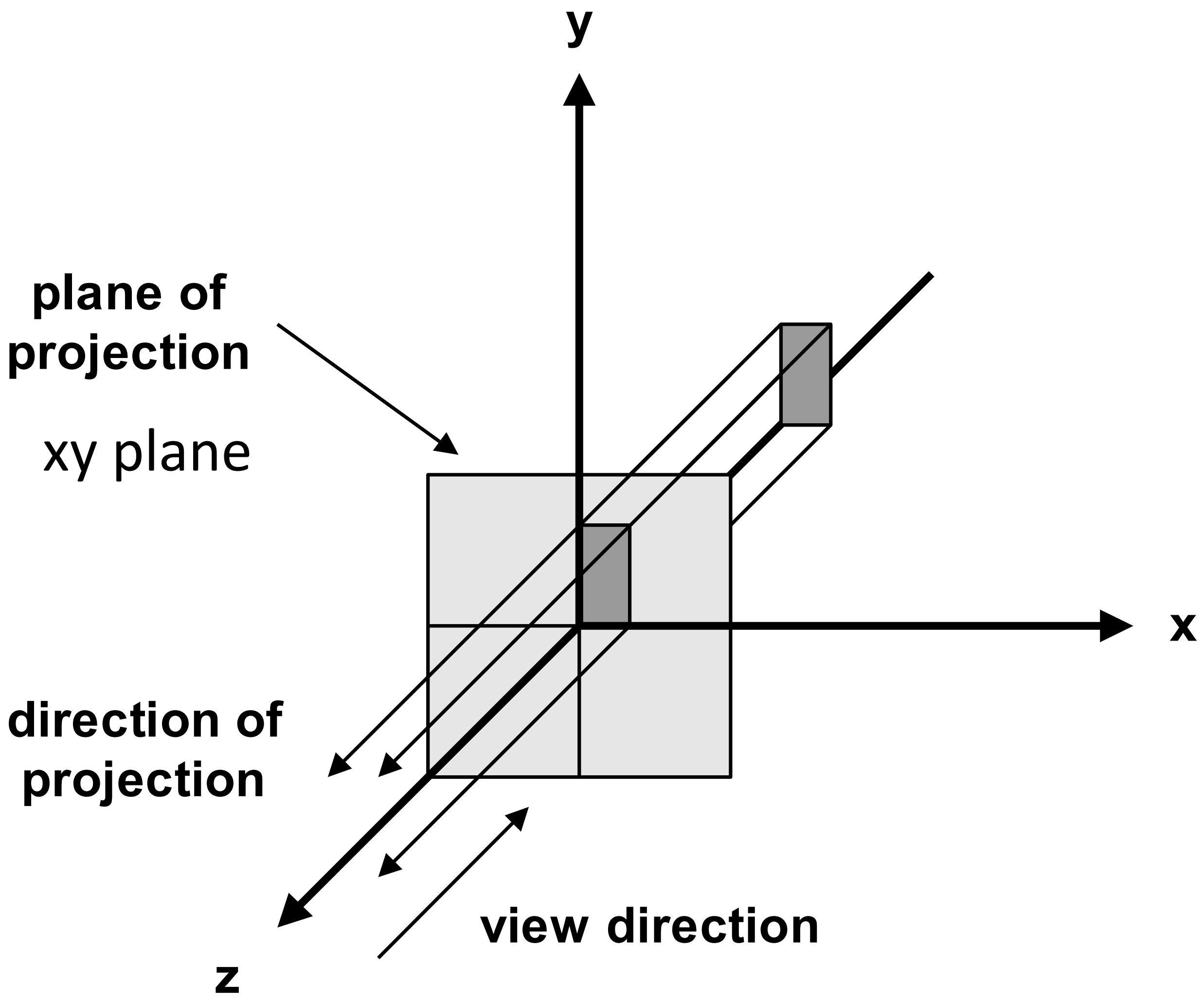
Finding Vanishing Point

- Find (a) the vanishing points for a given perspective transformation in the direction given by a vector U, (b) principal vanishing point.

- Family of parallel lines having the direction $U(u_1,u_2,u_3)$ can be written in parametric form as
  - $x = u_1t+p$, $y = u_2t+q$, $z = u_3t+r$
  - here P(p, q, r) is any point on the line

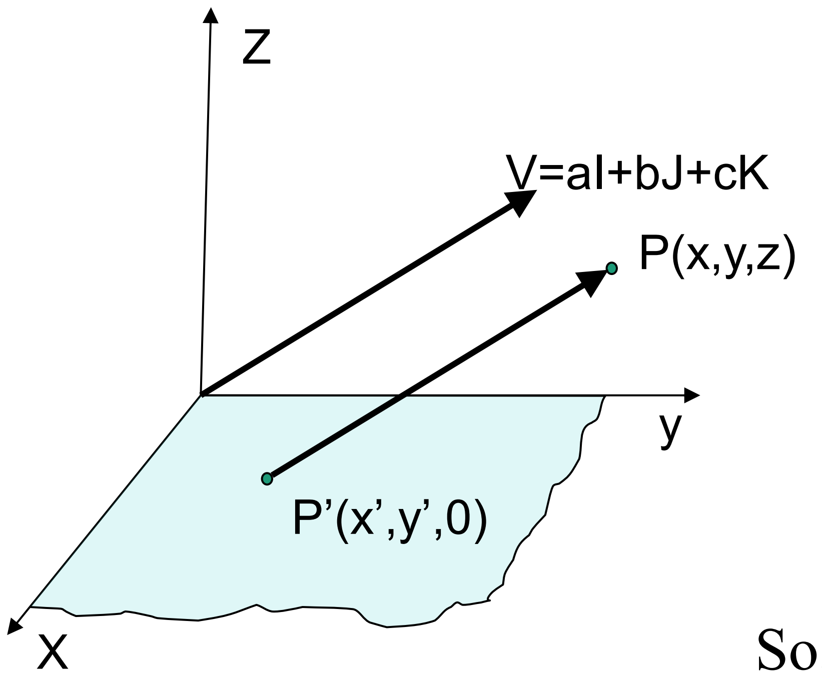
- Let, proj(x,y,z,1) = (x', y', z', h)
  - $x' = (d+an_1)(u_1t+p) + an_2(u_2t+q) + an_3(u_3t+r) – ad_0$
  - $y' = bn_1(u_1t+p) + (d+bn_2)(u_2t+q) + bn_3(u_3t+r) – bd_0$
  - $z' = cn_1(u_1t+p) + cn_2(u_2t+q) + (d+cn_3)(u_3t+r) – cd_0$
  - $h = n_1(u_1t+p) + n_2(u_2t+q) + n_3(u_3t+r) – d_1$

Orthographic Projection Matrix



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \\ 1 \end{pmatrix}$$

Parallel projection on xy plane with DOP $\mathbf{V} = a\mathbf{I} + b\mathbf{J} + c\mathbf{K}$



V and $\overline{P'P}$ has same direction,

so $\overline{P'P} = kV$

Comparing components

$$x - x' = ka \quad y - y' = kb \quad z - z' = kc$$

Since, projection on xy plane, $z' = 0$

So, $k = \dfrac{z}{c} \quad x' = x - \dfrac{a}{c}z \quad y' = y - \dfrac{b}{c}z$

So, $\quad P' \quad = \quad Par_V \bullet P$

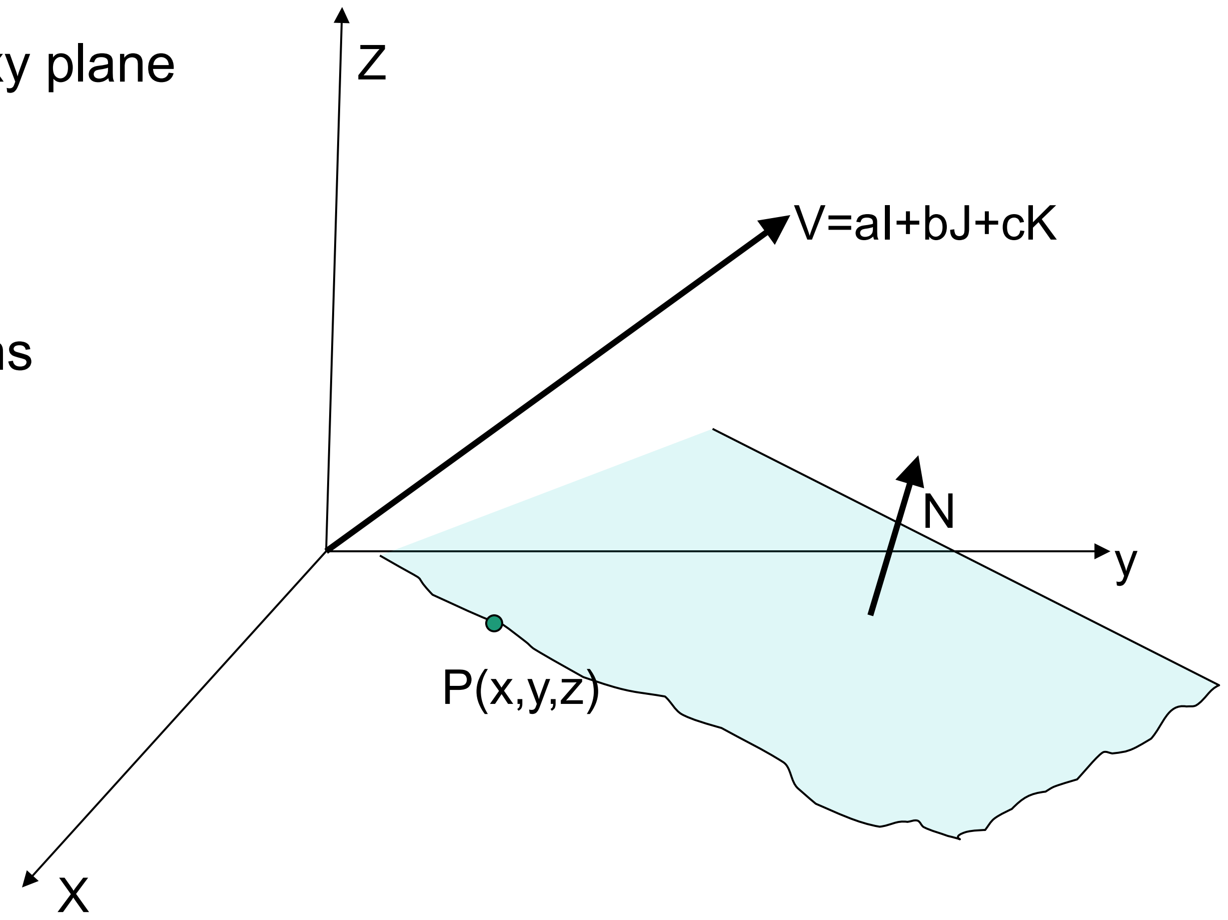
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} x - a/c\, z \\ y - b/c\, z \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -a/c & 0 \\ 0 & 1 & -b/c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Derive eq of parallel projection with DOP **V** = a**I** + b**J** + c**K** on plane with plane normal N, passing through P0

Do by yourself

Hint:

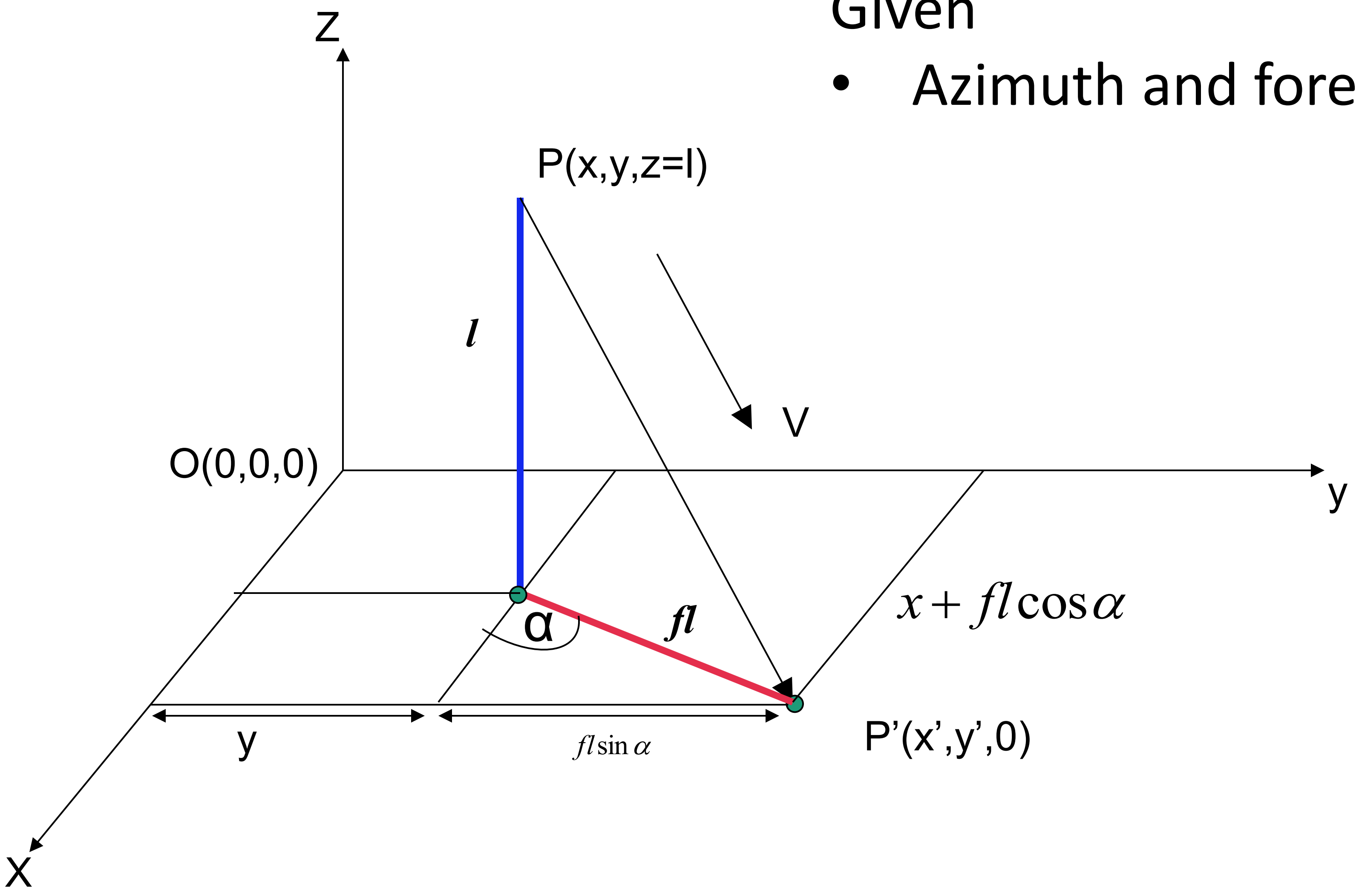
- Align the plane with xy plane
- Adjust the DOP
- Do the projection
- Do the reverse actions



Math of Oblique Projection

Given

- Azimuth and foreshortening factor

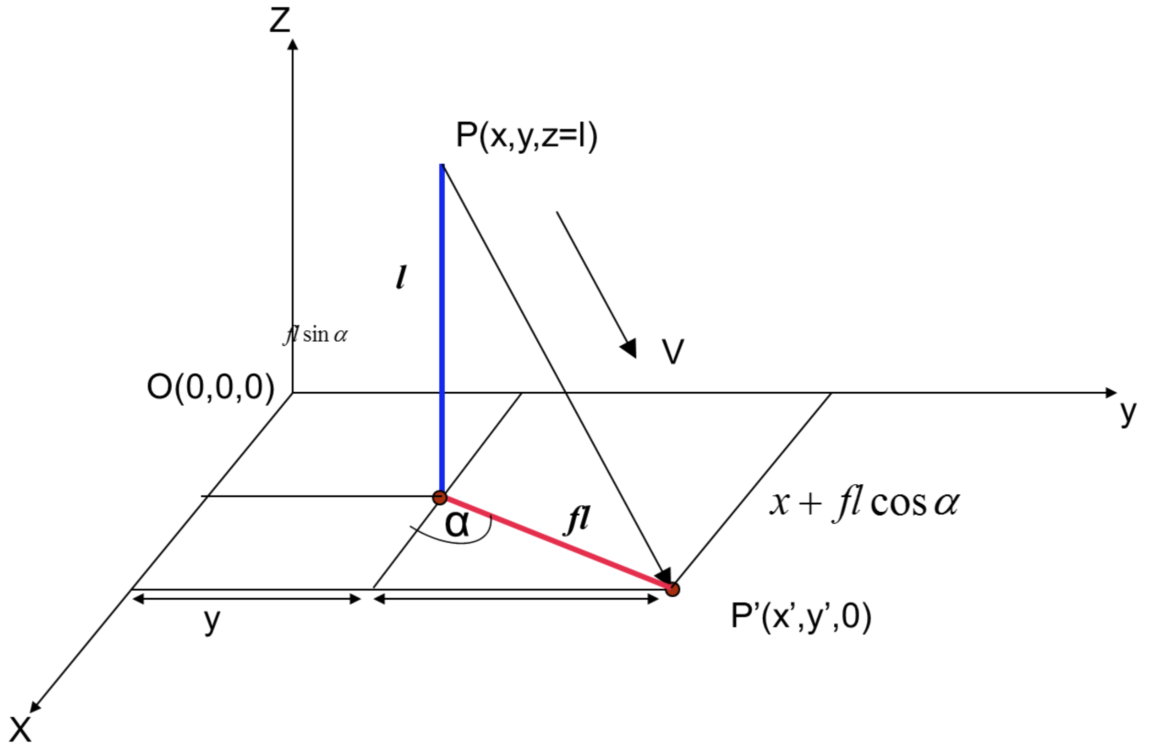


Solution

$$x' = x + fl\cos\alpha = x + zf\cos\alpha$$
$$y' = y + fl\sin\alpha = y + zf\sin\alpha$$
$$z' = 0$$

In Matix form

$$\text{Par}_v = \begin{bmatrix} 1 & 0 & f\cos\alpha & 0 \\ 0 & 1 & f\sin\alpha & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

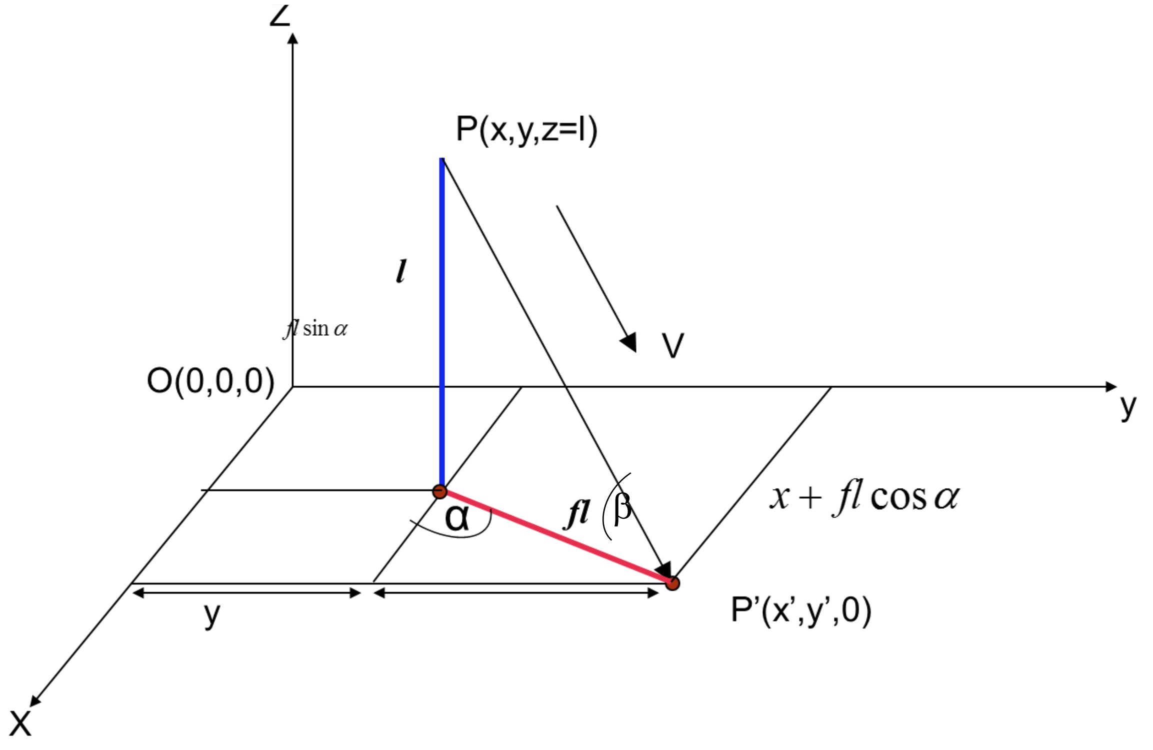


Solution

Given

- Azimuth and elevation

$$x' = x + fl\cos\alpha = x + zf\cos\alpha$$
$$y' = y + fl\sin\alpha = y + zf\sin\alpha$$
$$z' = 0$$



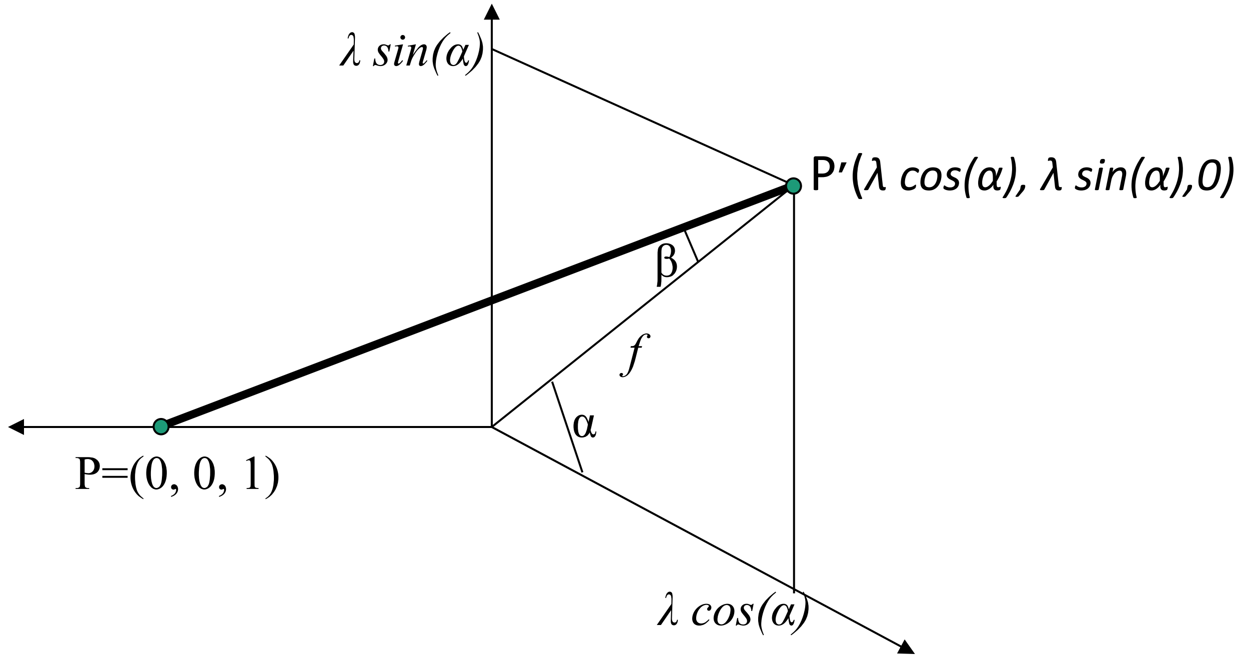
In Matix form

$$\mathrm{Par_v} = \begin{bmatrix} 1 & 0 & f\cos\alpha & 0 \\ 0 & 1 & f\sin\alpha & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$tan\beta = \frac{1}{f}$$

Parallel projection

- Cavalier, cabinet and orthographic projections can all be specified in terms of (α, β) or (α, f) since
  - $\tan(\beta) = 1/f$



Parallel projection

$\lambda=1$	$\beta = 45$	Cavalier projection	$\alpha = 0 - 360$
$\lambda=0.5$	$\beta = 63.4$	Cabinet projection	$\alpha = 0 – 360$
$\lambda=0$	$\beta = 90$	Orthogonal projection	$\alpha = 0 – 360$

Problems

- Schaums Series
  - 7.2
  - 7.6
  - A2.14